PREFERENCE NETWORKS IN MATCHING MARKETS

SAMIR CHOWDHURY

Abstract. This is a short note to sketch out the ideas for a talk given in CSE 5339: Network Data Analysis.

Market interactions between buyers and sellers form an interesting class of problems in network data analysis. To illustrate some of the major results in this field, we will consider three different models in which resources/objects are allocated to users. The three models correspond to three types of preferences: (1) binary (want/do not want), (2) weighted and visible, and (3) weighted and hidden. Here "weighted and visible" means that the users openly state their true valuations for the different objects, and "weighted and hidden" means that the users have a valuation that they internally decide, but what they state openly might be different.

The main ideas in this note are presented in [1].

1. Networks with binary preferences

Definition. A graph $G = (V, E)$ is *bipartite* if there exist $X, Y \subseteq V$ such that $X \sqcup Y = V$ and all $e \in E$ have the form $e = (x, y)$, where $x \in X$ and $y \in Y$. We will occasionally denote such graphs by $G = (X, Y, E)$.

We adopt the convention that two edges in a graph are *disjoint* if they do not have a common endpoint.

Definition. A *matching* in a bipartite graph $G = (X, Y, E)$ is a set $M \subseteq E$ consisting of disjoint edges. In the case where $|X| = |Y|$, a *perfect matching* is a matching M such that $\pi_X(M) = X$ and $\pi_Y(M) = Y$. Here $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the natural projections.

These minimal definitions allow us to ask the following question:

Question 1. *For what conditions does a bipartite graph admit a perfect matching?*

The preceding question admits a result known as Hall's Matching Theorem. This result is not just interesting from a combinatorial perspective; we now describe an application of this theorem to wireless sensor networks. Such a network consists of sensor nodes that are able to communicate with other nodes within a certain distance, and a sink node that acts as a hub. A sensor node can transmit information directly to the sink if it is close enough, but otherwise, it needs to transmit to another sensor node that is ideally closer to the sink. The objective of a routing protocol is to relay information from a sensor node to the sink node in as few hops as possible. The constraint of having a minimal number of hops is used to minimize the latency in the network.

In graph theoretical terms, this problem now becomes equivalent to finding a spanning tree rooted at the sink node. A further constraint would be to minimize the maximum degree of any node in this tree—this consideration is to minimize the energy usage of any node. Thus the problem can now be formulated as follows:

Question 2. *Given a graph and a sink node, find a spanning tree rooted at the sink node such that:*

Date: April 5, 2016.

(1) The maximum degree of any node in the tree is minimized, and
(2) Each edge connects a node to another node that is closer to the sink than itself.

Bokal et al solved this problem via a generalization of Hall's theorem [2]. The idea is that because one does not want to traverse between nodes at the same distance from the sink, these nodes can be viewed as belonging to one partition of a bipartite graph, whereas the other partition consists of nodes that are closer to the sink node. Finding a perfect matching thus allows one to satisfy both conditions in Question 2.

Before formally stating Hall's theorem, we need one more definition:

Definition. Let $G = (X, Y, E)$ be a bipartite graph. A set $S \subseteq X$ (resp. $S \subseteq Y$) is *constricted* if $|\pi_Y(\pi_X^{-1}(S))| < |S|$. Alternatively, S is constricted if its neighbor set $N(S)$ satisfies $|N(S)| < |S|$. Here $N(S)$ is defined as:

$$N(S) = \{y \in Y : x \in S, (x, y) \in E\}.$$

Theorem 1 (Hall's Matching Theorem, 1935)**.** *Let $G = (X, Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then there exists a perfect matching $M \subseteq E$ if and only if G does not contain any constricted sets.*

Hall's theorem is equivalent to another theorem attributed to König and Egerváry. Berge and Tutte also have related theorems. We do not discuss those results here, but point the reader towards notes by Wildstrom on the subject [3].

Finally, we remark that a *preference network* is any bipartite graph where one partition consists of *users*, the other to *products/services*, and the edges correspond to preferences.

2. Networks with visible weighted preferences

Suppose we now have a bipartite graph consisting of *buyers* and *sellers*, where each seller has an item for which they wish to obtain a particular price. We view this as a weighted graph $G = (B, S, E, \omega)$, where $|B| = |S|$, and the weights ω are defined as follows:

- $\omega(b_i, s_j) = v_{ij}$, the valuation of seller j's item by buyer i,
- $\omega(b_i, b_i) = 0$ for all i, and
- $\omega(s_i, s_i) = p_i$, the price of seller i's item.

Occasionally, a buyer may be willing to pay more for a seller's item than the price set by the seller. The *payoff* of buyer i for seller j's item is given by $v_{ij} - p_i$. The *preferred seller* of buyer b_i is the seller j for whom b_i's payoff is maximized. Note that a buyer may have multiple preferred sellers. We also stipulate that if a buyer is unable to obtain a nonnegative payoff for any seller, then the buyer will simply be better off not transacting. Finally, given a set of valuations and prices, the *preferred seller graph* is simply the graph of buyers and sellers consisting of edges connecting a buyer to their preferred seller(s).

In an ideal situation, each buyer would have a unique preferred seller, and no two buyers would prefer the same seller. Then the buyers would walk away with the items that maximize their payoffs, without having any conflicts with each other. It may also be the case that some buyers have multiple overlapping preferred sellers—in this case, the tie could be broken with some coordination between the buyers, and they would still be able to maximize their payoffs. A set of prices for which either of these two situations holds is called a set of *market clearing prices*. With the definitions we have already introduced, we can equivalently say that a set of prices is market clearing if the corresponding preferred seller graph contains a perfect matching. We illustrate this situation in Figure 1.

One may ask what the conditions on the valuations and prices needs to be for a set of market clearing prices to exist. A result of Egerváry, also known as the Hungarian method, gives an answer to this problem.

Theorem 2 (Egerváry, 1918)**.** *Given a graph (B, S, E, ω) and a set of buyer valuations, there always exists a set of market clearing prices.*

This theorem allows for the following simple way of assigning items to buyers: given a set of buyer valuations, check for a constricted set in the preferred seller graph. If there is none, then a perfect matching can be found, and we are done. If there is a constricted set $S \subseteq B$, then raise the prices of the items in $N(S)$ by one unit, and repeat the preceding steps. Egerváry proved that this process always terminates, i.e. we can obtain a preferred seller graph containing a perfect matching by successively increasing prices.

Remark 3. Market-clearing prices are beneficial to buyers by construction, because they allow the buyers to maximize their individual payoffs. But they are also beneficial to the sellers, in the sense that they maximize the amount of money that is changing hands. To see why this is true, let M be a perfect matching in a preferred seller graph corresponding to a set of market clearing prices. Then we have:

$$\begin{aligned}\text{Total payoff to buyers} &= \sum_{(i,j)\in M} (v_{ij} - p_j) \\ &= \sum_{(i,j)\in M} v_{ij} - \sum_{j} p_j.\end{aligned}$$

Since the prices are fixed, it follows that the total payoff is maximized whenever the total valuation is maximized. Since we know that the total payoff is being maximized, it follows that the total valuation, i.e. the total amount of money in the market, is also maximized.

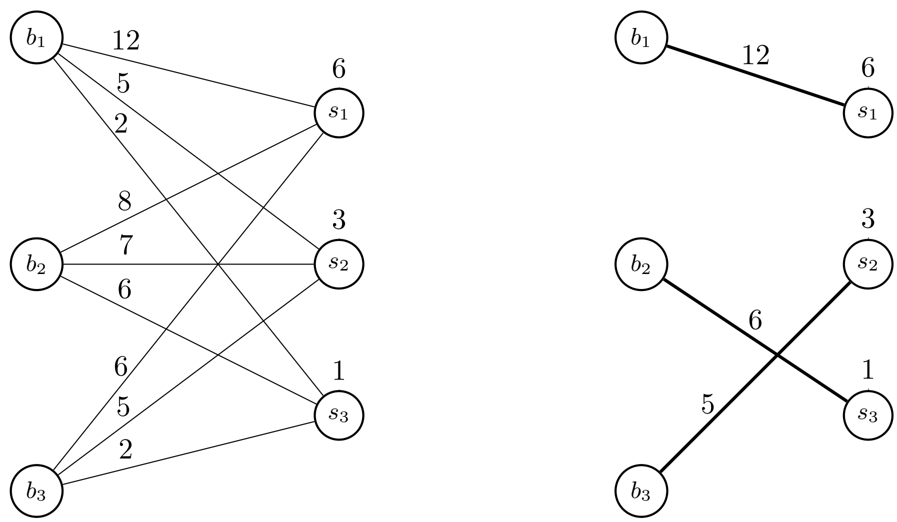

FIGURE 1. In the figure on the left, a set of buyers propose valuations for a set of items. On the right, a perfect matching in the preferred seller graph is displayed.

3. NETWORKS WITH HIDDEN WEIGHTED PREFERENCES

We now analyze a related situation which can be modeled by a preference network. Internet search engines generate a significant portion of their revenue by auctioning off advertisement slots. Such an interaction can be modeled by a bipartite graph (A, S, E, ω) where A consists of *advertisers*, S consists of advertising *slots*, and we assume $|A| = |S|$ for convenience. The advertising slots are simply positions on a search result page where ads are displayed, with the top position being the most desirable.

Note that if the search engine knows the valuations of all the advertisers for each of the slots it offers, then it can simply set up market clearing prices (via Egerváry's theorem). However, if the true valuations of the advertisers are not known, then the search engine needs to devise a method that encourages the advertisers to employ "truthful bidding", i.e. bidding precisely the amount of their internal valuations.

One might ask what the consequences of false bidding might be. In this situation, an advertiser a_i has a particular valuation v_{ij} for slot s_j, but he bids a price $v_{ij} - \varepsilon$. If nobody else bids for this slot, then a_i wins and makes a profit, whereas the search engine loses some of its potential revenue. But what typically happens is that a_i starts off with a bid $v_{ij} - \varepsilon$ for large ε, hoping to make more of a profit. Seeing this low bid, other advertisers are encouraged to bid slightly higher. This process runs continuously—and in the worst case, can lead to a highly unstable market where the bids are updated constantly and both the advertisers and the search engine are forced to expend resources unnecessarily to update their prices continuously.

It turns out that there exists a method, known as the *Vickrey-Clarke-Groves (VCG) principle*, which enforces a situation where the best strategy for an advertiser is to bid their true valuation for each slot. The VCG principle is a generalization of a certain auction method known as second-price auction. We will not go into details on how the VCG principle is applied to matching markets; instead, we'll look at its original application to second price auctions.

3.1. **First price and second price auctions.** In a first price auction, the highest bidder wins, and has to pay the winner's bid. Suppose a bidder b has an internal valuation v for an item s. Then b has three bidding options:

- Bid less than v: this leads to a possibility of making some profit, with the caveat that another bidder will win
- Bid greater than v: this is a lose-lose case, because b would have to pay a value greater than v even for a win
- Bid exactly equal to v: then b will not lose or gain any value.

In a first price auction, the optimal strategy for b is always to bid a value slightly lower than v, as this potentially leads to a profit.

In a second price auction, the highest bidder wins, but has to pay only the second-place bid. Suppose now there are bidders b_i, b_j with internal valuations $v_i < v_j$ for an item s. Then the following cases can occur:

- b_j bids a value $v'_j = v_j + \varepsilon$. Then b_j still has to pay only what b_i bids; in case b_i bids a value $v'_i > v_j$, then b_j can still win, but will have to pay more than intended.
- b_j bids v_j. Then b_j has to pay at most v_i (note that b_i would not rationally bid higher than v_i), and profits by $v_j - v_i$.
- b_j bids a value $v''_j = v_j - \varepsilon$. Then b_j can only make a profit if b_i bids a value $v''_i < v''_j$, but otherwise will lose the auction whereas a win was possible with a non-negative payoff.

By analyzing the different cases, it is possible to see that truthful bidding is indeed the optimal strategy in a second price auction.

The VCG principle generalizes the second price auction to include multiple items. Once the VCG principle has guaranteed that the buyers/advertisers are placing truthful bids, the problem of finding market clearing prices can be solved using the methods described in the previous section.

Remark 4. In practice, Google uses an auction method called *Generalized Second Price (GSP) auction*. This differs from the VCG method in that bidding true values is no longer an optimal strategy, and it is also thought to be better at increasing revenue for the search engine than VCG. However, there are trade-offs between both methods, and fully exploring the connections between the two, along with revenue comparison, seems to be an open problem for now.

References

[1] Easley, David, and Jon Kleinberg. Networks, crowds, and markets: Reasoning about a highly connected world. Cambridge University Press, 2010.
[2] Bokal, Drago, Boštjan Brešar, and Janja Jerebic. "A generalization of Hungarian method and Halls theorem with applications in wireless sensor networks." Discrete Applied Mathematics 160.4 (2012): 460-470.
[3] http://aleph.math.louisville.edu/teaching/2010SP-682/notes-100121.pdf