Brenier's polar factorization theorem and McCann's generalization

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- Theorem (Brenier's factorization theorem)

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$$s = t \circ u, \text{ where}$$

$$u : \Omega \to \Omega \text{ is a volume preserving map and}$$

$$t = \nabla\psi : \mathbb{R}^n \to \mathbb{R}^n \text{ is the gradient of a convex function}$$

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  - **Question:** What is the relation between this and optimal transport?
  - **Answer:** Proof depends on the solution of Monge-Kantorovich problem.

Monge problem

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- Monge problem is finding the cost minimizing transport map G.
- Existence of the solution depends on properties c. In this presentation we assume that M is a metric space and $c = d^2/2$.

Kantorovich problem

- M, μ, ν, c as above. Let $p, q : M \times M \to M$ denote the projection onto the first coordinate and second coordinate respectively. The set of all transport plans from μ to ν is defined by

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Relation between Monge and Kantarovich Problem

- Kantorovich problem is a relaxation of the Monge problem in the following sense:
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- The image of the map above is the set of measures in $\Gamma(\mu,\nu)$ whose support is a graph.
- $\Gamma(u,v)$ is a convex subset of a Banach space (i.e. dual space of the continuous functions $(C(M \times M), l_\infty)$). This is helpful for showing the existence and uniqueness of solutions.

Existence of Monge solutions, uniqueness of Kantorovich solutions

Let M be an n-dimensional connected compact Riemannian manifold, and μ, ν be Borel measures on M. Then there is a convex potential function $\psi : M \to \mathbb{R}$ such that

- $G(x) := \exp_x(\nabla\psi)$ is a transport map.
- G is the only transport map arising this way. It solves Monge's problem.
- Kantorovich problem has a unique solution.
- Kantorovich problem is obtained from G.

McCann's Factorization Theorem

Let M be a connected compact Riemannian manifold. Let $s : M \to M$ be a Borel map which never maps positive volume into zero volume. Then s factors uniquely into the form $s = t \circ u$, where

$$u : M \to M \text{ is a volume preserving map and}$$

$$t = \exp(\nabla\psi) : M \to M$$

where ψ is a convex function $\psi : M \to \mathbb{R}$.

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- Let t be the solution of the Monge problem $S(\mu, \nu)$ arising from the potential $\psi : M \to \mathbb{R}$.
- Let t^* be the solution of the Monge problem $S(\nu, \mu)$. Show that t, t^* are inverses almost everywhere. Let $u = t^* \circ s$.
- Then $t \circ u = s$, μ almost everywhere. Furthermore $u_*(\mu) = t^*_* s_*(\mu) = t^*_*(\nu) = \mu$, hence u is measure preserving.

- McCann,R, J. *Polarization of maps on Riemannian Manifolds.* Max-Planck-Institut fur Mathematik in den Naturwissenschaften Leipzig, Preprint no: 41, 1999.
- McCann, Robert J., and Nestor Guillen. *Five lectures on optimal transportation: geometry, regularity and applications.* Analysis and geometry of metric measure spaces: lecture notes of the séminaire de Mathématiques Supérieure (SMS) Montréal (2011): 145-180.