Gromov-Hausdorff distances in Euclidean spaces

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Abstract

The purpose of this paper is to study the relationship between measures of dissimilarity between shapes in Euclidean space. We first concentrate on the pair Gromov-Hausdorff distance (GH) versus Hausdorff distance under the action of Euclidean isometries (EH). Then, we (1) show they are comparable in a precise sense that is not the linear behaviour one would expect and (2) explain the source of this phenomenon via explicit constructions. Finally, (3) by conveniently modifying the expression for the GH distance, we recover the EH distance. This allows us to uncover a connection that links the problem of computing GH and EH and the family of Euclidean Distance Matrix completion problems. The second pair of dissimilarity notions we study is the so called L^p-Gromov-Hausdorff distance versus the Earth Mover's distance under the action of Euclidean isometries. We obtain results about comparability in this situation as well.

1. Introduction

The problem of shape/object matching/comparison appears in various disciplines. The problem has been approached by several different techniques in the past 20 years. Recently in [18, 19], the use of the Gromov-Hausdorff (referred to as GH from now on) distance (as a measure of shape similarity) has been proposed as a way of formalizing and providing new tools for tackling the problem of shape matching under certain invariances. We refer the reader to [19] for an account. One of the main features of the GH approach is that shapes should be regarded as *metric spaces* where the metric with which the shapes are endowed depends on the type of invariance one wishes to consider. For example, if one is interested in invariance to Euclidean isome-

tries, it is natural to endow shapes with the Euclidean metric. However, if one is interested in invariance to the so called *bends* (deformations of a surface that preserve the geodesic metric) then one should augment the shapes/surfaces with their Riemannian geodesic distance to thus form metric spaces. The similarity between two shapes is therefore measured as a similarity between the metric spaces one defines from the shapes.

The main purpose of this paper is to present and prove some desirable theoretical properties of the GH distance when the class of shapes one works with are subsets of Euclidean space. Some of these properties give rise to some new practical procedures based on these ideas.

Under the assumption that the metric spaces we want to compare are *Euclidean*, we will be interested in proving the *equivalence* of the GH distance with the metric obtained from the Hausdorff distance that takes quotient with all Euclidean isometries (EH henceforth). Roughly, the EH distance attempts to find the optimal Euclidean isometry that aligns the two shapes (in Euclidean space) under the Hausdorff distance.[1] We prove important and interesting results about this connection. Typically, the EH is approximately implemented via the Iterative Closest Point algorithm (ICP), [23, 21].

The main difference between the GH and EH distances lies in that while the former only looks at the interpoint distance between points on each shape, *without any regard for the ambient space*, the latter requires finding a *Euclidean isometry*, meaning an isometry in ambient space, that aligns the shapes. From this simple observation it is almost obvious that the EH distance should furnish an upper bound for the GH distance. In this paper we prove that this bound and a bound in the opposite direction both hold.

Given the plethora of methods available for shape/object matching, we believe that, in order to

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[1]Precise definitions are given in §3.

obtain a deep understanding of the problem of shape matching and find possible avenues of improvement, it is of extreme importance to discover and establish relations between these methods. Theoretical understanding of these methods will lead to expressing conditions of validity of each approach or family of approaches. This can no doubt help in (a) guiding the choice of which method to use in a given practical application, (b) telling what parameters (if any) should be used for the particular chosen method, and (c) clearly determining what are the *guarantees* of this particular method for the task at hand. This paper is in the same vein as [17] and tries to establish more connections between different approaches. This is an overview of our main results:

(A) We observe (this is well known in the metric geometry community) that, in general, the GH distance between two Euclidean metric spaces does not agree with the EH distance. It is true, however, that for any pair of Euclidean shapes,[2] the GH distance is bounded above by the EH distance:

$$\mathrm{GH} \leqslant \mathrm{EH}.$$

(B) We prove that the EH distance admits (an increasing function of) the GH distance as an upper bound, more precisely that, for some constants $c > 0$,

$$\mathrm{EH} \leqslant c \cdot \mathrm{GH}^{\frac{1}{2}}.$$

This result is based on a Theorem by Alestalo et al., [2]. This, taken together with the previous item, proves that the GH and EH distances are *equivalent* (or comparable). We discuss how the exponent can be upgraded to 1 by restricting us to a certain class of shapes. One consequence of these bounds is that the numerical techniques of [19, 7], which compute approximations to the GH distance, could, in principle, also be used for tackling the problem of matching shapes under invariance to Euclidean isometries.

(C) We show how to obtain similar bounds for the L^p-Gromov-Hausdorff (GHp) ([17]) and Euclidean isometries invariant Wasserstein (or Earth Mover's) distances (EWp), [9]. This type of bounds are sometimes implicitly used in the proofs of correctness of registration algorithms ([12] §4). We prove in this case that

$$\mathrm{GHp} \leqslant \mathrm{EWp} \leqslant c \cdot \mathrm{GHp}^{\frac{1}{4}}.$$

As a byproduct, we obtain a quantitative statement (Lemma 2) that supports the idea that the L^p-GH distances of [17] are well adapted for tackling the problem of *partial shape matching*.

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[2]By this we mean that the shapes are subsets of Euclidean space, and are made into metric spaces by endowing them with the Eulidean metric.

(D) By using one of the many equivalent expressions for the GH distance, we identify a modification of this expression that transforms the GH distance into the EH distance, thus "closing the gap" between the two. This formula we identify is therefore an equivalent expression for the EH distance. By doing this we uncover a relationship between the EH distance and the so called Euclidean Distance matrix completion problem (EDMCP), [1, 3]. We also close the gap in the case of the Wasserstein type of distances. This allows us to propose a new algorithmic idea based on a certain L^2-GH distance introduced in [17] that could potentially be useful for solving some Euclidean isometry problems.

We believe that the material in this paper will further the understanding of the shape matching problem, by exposing more links between the different approaches that have been proposed in the community and by providing new tools for analyzing these approaches.

Due to space constraints we do not present the proofs of some of our results. These will be provided elsewhere. The proof of Theorem 4 is, however, presented. It contains some of the elements used for proving the results whose proofs we omit.

2. The Gromov-Hausdorff distance

The features and advantages of using the GH distance for shape matching have been discussed by [19], we refer the reader to that publication for further details.

The main point the reader should keep in mind is that shapes are regarded as *metric spaces*. We concentrate on the main definitions needed for our exposition. A good general reference for metric spaces, basic point set topology and the GH distance is [8].

Now, we will introduce the Hausdorff distance, followed by the original definition of the GH distance. Other equivalent definitions of the GH distance that are necessary for our argument will also be presented.

Definition 1 (Hausdorff distance). *Let (Z, d) be a compact metric space and A, B two closed subsets of Z. One defines the Hausdorff distance between A and B to be*

$$d_{\mathcal{H}}^Z(A, B) := \max \left(\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right). \quad (1)$$

Following [14], we introduce the Gromov-Hausdorff distance between (compact) metric spaces X and Y:

$$d_{\mathcal{GH}}(X, Y) := \inf_{Z, f, g} d_{\mathcal{H}}^Z(f(X), g(Y)) \quad (2)$$

where $f : X \to Z$ and $g : Y \to Z$ are *isometric embeddings* (distance preserving) into the metric space (Z, d).

This expression seems daunting from the computational point of view. We will recall equivalent, tamer, expressions below. The following diagram depicts the general construction:

$$(X, d_X) \overset{d_{\mathcal{GH}}}{\Longleftrightarrow} (Y, d_Y) \qquad (3)$$
$$f \downarrow \qquad\qquad \downarrow g$$
$$(f(X), d) \overset{d_{\mathcal{H}}^Z}{\Longleftrightarrow} (g(Y), d)$$

Despite its apparent complexity, as was already pointed out in [19], expression (2) helps to cast the procedure of [10] inside the Gromov-Hausdorff realm.

Definition 2 (Correspondence)**.** *For sets A and B, a subset $R \subset A \times B$ is a* correspondence *(between A and B) if and and only if*
(1) $\forall\, a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
(2) $\forall\, b \in B$, there exists $a \in X$ s.t. $(a, b) \in R$.
Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B.

Consider metric spaces (X, d_X) and (Y, d_Y). Let $\Gamma : X \times Y \times X \times Y \to \mathbb{R}^+$ be given by $(x, y, x', y') \mapsto |d_X(x, x') - d_Y(y, y')|$. Then, the Gromov-Hausdorff distance between X and Y can be rewritten as

$$d_{\mathcal{GH}}(X, Y) := \frac{1}{2} \inf_{R \in \mathcal{R}(X,Y)} \max_{\substack{(x,\,y)\,\in\,R \\ (x',\,y')\,\in\,R}} \Gamma(x, y, x', y'). \quad (4)$$

Definition 3 (Metric Coupling)**.** *From now on let $\mathcal{D}(d_X, d_Y)$ denote the set of all possible* metrics *on the disjoint union of X and Y, $X \sqcup Y$. Let $d \in \mathcal{D}(d_X, d_Y)$. This means that d, besides satisfying all triangle inequalities, it also satisfies that $d(x, x') = d_X(x, x')$ and $d(y, y') = d_Y(y, y')$ for all $x, x' \in X$ and $y, y' \in Y$.*

Remark 1. *One can equivalently (in the sense of equality) define the Gromov-Hausdorff distance between metric spaces (X, d_X) and (Y, d_Y) as ([8] pp. 255)*

$$d_{\mathcal{GH}}(X, Y) = \inf_d\, d_{\mathcal{H}}^{(X \sqcup Y, d)}(X, Y) \qquad (5)$$

where the infimum is taken over $d \in \mathcal{D}(d_X, d_Y)$. This expression is central to our presentation.

Remark 2. *It was pointed out in [16] that (4) can be recast in a somewhat clearer form: For functions $\phi : X \to Y$ and $\psi : Y \to X$ consider the numbers $A(\phi) := \sup_{x_1,x_2 \in X} |d_X(x_1, x_2) - d_Y(\phi(x_1), \phi(x_2))|$, $B(\psi) := \sup_{y_1,y_2 \in Y} |d_X(\psi(y_1), \psi(y_2)) - d_Y(y_1, y_2)|$ and $C(\phi, \psi) := \sup_{x \in X,\, y \in Y} |d_X(x, \psi(y)) - d_Y(\phi(x), y)|$, then*

$$d_{\mathcal{GH}}(X, Y) = \inf_{\substack{\phi\,:\,X\,\to\,Y \\ \psi\,:\,Y\,\to\,X}} \frac{1}{2} \max\left(A(\phi), B(\psi), C(\phi, \psi)\right). \quad (6)$$

This expression is the central idea behind the computational approaches of [19] and [7].

Below we state many well known properties of the GH distance:

Proposition 1 ([8] Ch. 7 and [16])**.** *Let X and Y be two compact metric spaces. Then the following assertions are true:*

1. *The following equalities hold: (2)=(4)=(5)=(6).*

2. *The GH distance is a* true metric *on the set of classes of isometric metric spaces.*

3. *If $d_{\mathcal{GH}}(X, Y) < \delta$ then there exist $f : X \to Y$ such that $A(f) < 2\delta$ and $f(X)$ is a 2δ-net of Y.*

4. *The GH distance is bounded: $d_{\mathcal{GH}}(X, Y) \leqslant \frac{1}{2} \max\left(\mathbf{diam}\,(X), \mathbf{diam}\,(Y)\right)$.*

Remark 3. *Note that via (4) it is easy to see that $d_{\mathcal{GH}}(X, \{p\}) = \mathbf{diam}\,(X)/2$. In fact, if $Y = \{p\}$, then $d_Y(y, y') = 0$ for all choices of y, y' and hence $\Gamma(x, y, x', y') = d_X(x, x')$ for all $x, x' \in X$ and $y, y' \in Y$. Note that $\mathcal{R}(X, Y)$ consists of a unique correspondence $R = \{(x, p) | x \in X\}$. Then $d_{\mathcal{GH}}(X, Y) = \frac{1}{2} \max_{x,x'} d_X(x, x')$.*

3. The case of Isometries in Euclidean Space

Let $E(n)$ denote the Euclidean group of n-dimensional Euclidean space.[3] Consider the distance between compact subsets X, Y of $\mathbb{R}^n$ given by:

$$d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) := \inf_{T \in E(n)} d_{\mathcal{H}}^{\mathbb{R}^n}\left(X, T(Y)\right). \quad (7)$$

It is easy to check that:

Proposition 2. $d_{\mathcal{H},iso}^{\mathbb{R}^n}(,)$ *is a metric on the set of isometry classes of compact subsets of Euclidean space $\mathbb{R}^n$.*

This notion of distance has received a lot of attention by researchers. It is frequently approximated by using the Iterative Closest Point technique (ICP).[4] See for example [13, 15, 23, 20] and references therein.

Assume for simplicity that X and Y are finite: $X = \{x_1, \ldots, x_\ell\}$ and $Y = \{y_1, \ldots, y_m\}$. It follows from the fact that GH is a *metric* on (the isometry classes of) metric spaces (Proposition 1) that if $d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|)) = 0$ then there exists an *isometric transformation* that maps X into Y. This simply

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[3]Recall that its elements are all Euclidean isometries: translations, rotations and reflections and linear combinations of these.

[4]ICP techniques actually deal with a certain L^2 version of (7).

and merely means that (a) $\ell = m$ and (b) that there exist a permutation π of $\{1, \ldots, m\}$ s.t.

$$\|x_i - x_j\| = \|y_{\pi_i} - y_{\pi_j}\| \tag{8}$$

for all $i, j = 1, \ldots, m$.

Via Lemma 1 below, this condition allows us to construct an isometry T of the **ambient space** $\mathbb{R}^n$ such that $T(x_i) = y_{\pi_i}$ for all $i = 1, \ldots, m$.

Notice that (8) gives information only about correspondence between (finitely many) points of $\mathbb{R}^n$ and yet, we are able to extrapolate the map given by π into a full isometry of $\mathbb{R}^n$ into itself. The existence of such Euclidean isometry immediately implies that, also, $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) = 0$.

Lemma 1 (Folklore Lemma, [5]). *Let $p_1, \ldots, p_m$ and $q_1, \ldots, q_m$ be points in $\mathbb{R}^n$. If $\|p_i - p_j\| = \|q_i - q_j\|$ for every $i, j = 1, \ldots, m$, then there exists a Euclidean isometry T such that $T(p_i) = q_i$, for every $i = 1, \ldots, m$.*

Corollary 1. *If X and Y are compact subsets of $\mathbb{R}^n$ such that when endowed with the Euclidean metric they are isometric, then there exists an Euclidean isometry $T : \mathbb{R}^n \to \mathbb{R}^d$ such that $T(X) = Y$.*

A natural question to ask is whether it is still true that when all we know is that for some $\varepsilon \geqslant 0$, $d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|)) \leqslant \varepsilon$, then this implies the existence of a Euclidean isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d_{\mathcal{H}}^{\mathbb{R}^n}(X, T(Y)) \leqslant C \cdot \varepsilon^t \tag{9}$$

for some constant C and $t > 0$. This leads to searching for an upper bound for $d_{\mathcal{H},iso}^{\mathbb{R}^n}(,)$ in terms of $d_{\mathcal{GH}}(,)$, at least for a certain class of shapes. This is one of the most important points in the paper. The fact that in the case of Euclidean sets, just from the intrinsic similarity information provided by $d_{\mathcal{GH}}(X, Y)$ (i.e. no reference to an ambient space), we are able to construct an *ambient space isometry* T that renders X close to $T(Y)$ is nontrivial. We delve into this important question next.

3.1. Relating $d_{\mathcal{GH}}(,)$ with $d_{\mathcal{H},iso}^{\mathbb{R}^n}(,)$

We easily have:

Proposition 3. *For all compact $X, Y \subset \mathbb{R}^n$ one has*

$$d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|)) \leqslant d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y).$$

As before, let X and Y denote compact sets of $\mathbb{R}^d$. The first question one must answer is whether $d_{\mathcal{GH}}((X, \|\cdot\|), (Y, \|\cdot\|)) = d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y)$ in general.

This cannot be true, in general, as the following simple counterexample shows: Let $X = \{a, b, c\}$ where a, b, c are the vertices of an equilateral triangle of side lenght 1. Let $Y = \{p\}$. Then, it is clear that $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) = \sqrt{3}/3$ whereas, by Remark 3, $d_{\mathcal{GH}}(X, Y) = 1/2$. The reason for this is well known, the metric space Z into which the embedding is optimal for the GH distance (according to expression (2) of GH) can be thought of as a metric space with $-\infty$ curvature,[5] see Figure 2. This can also be interpreted via expression (5) of the GH distance by saying that the optimal metric is *not Euclidean*. We see this in detail in §3.2.

This can be extended in the following way: For a natural number $k < n$ consider X to be the $(k-1)$-simplex Δ_k consisting of k points all at unit distance from eachother, and $Y = \{p\}$. In this case, $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y)$ equals the *circumradius* of the simplex, which is known to be $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) = \sqrt{\frac{k}{2(k+1)}}$. But, still, $d_{\mathcal{GH}}(X, Y) = 1/2$. Thus, there can exist no constant C independent of dimension s.t. $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) = C \cdot d_{\mathcal{GH}}(X, Y)$ for all $n \in \mathbb{N}$ and Euclidean metric spaces X and Y in $\mathbb{R}^n$.

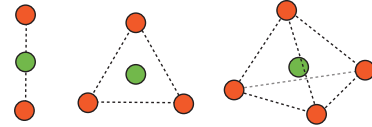Figure 1. *Simplices Δ_1, Δ_2 and Δ_3. Each of these have GH distance $1/2$ to the metric space composed of a single point. In green, we show the location of the circumcenter. The EH distances between each of the simplices to a single point are $1/2$, $\sqrt{3}/3$ and $\sqrt{3/8}$, respectively.*

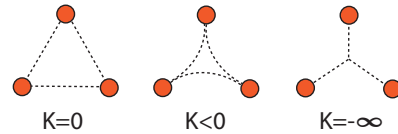Now we try to figure out what would be the maximal

Figure 2. *Embeddings of the simplex Δ_2 into spaces of decreasing curvature. Left to right, curvatures are 0, < 0 and $-\infty$. The latter picture corresponds to a tree-like metric space. In this case, the sides of the triangle branch off a central point and the distance of each vertex to this point is $\frac{1}{2}$. Thus, the optimal location of the embedding of $\{p\}$ corresponds to this branching point and hence the result $d_{\mathcal{GH}}(\Delta_2, \{p\}) = 1/2$.*

value that the exponent t in (9) can take. We adapt a construction from [2]. We claim that for all $\varepsilon \in (0, \frac{1}{4})$

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[5]The precise terminology for this is that this optimal embedding space is 0-*hyperbolic* in the sense of Gromov, see [8].

there exists finite sets $X, Y \in \mathbb{R}^2$ with

$$d_{\mathcal{GH}}(X,Y) \leqslant \varepsilon \quad \text{and} \quad d_{\mathcal{H}}(X, S(Y)) \geqslant \sqrt{\varepsilon/2}$$

for all Euclidean isometries $S : \mathbb{R}^2 \to \mathbb{R}^2$. Indeed, let $\{e_1, e_2\}$ denote the canonical basis of $\mathbb{R}^2$. Let $X = \{(0,0), \frac{e_1}{2}, e_1\}$ and $Y = \{(0,0), \frac{e_1}{2} + \sqrt{2\varepsilon} \cdot e_2\}$. Clearly, $2d_{\mathcal{GH}}(X,Y) \leqslant \sqrt{\frac{1}{4} + 2\varepsilon} - \frac{1}{2} \leqslant 2\varepsilon$. See Figure 3 for the construction. Let S be s.t. $d_{\mathcal{H}}^{\mathbb{R}^2}(Y, S(X)) = \alpha$. Note
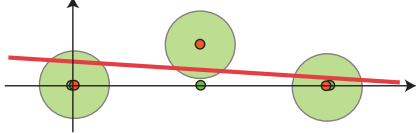

Figure 3. *Example that shows that $t = 1$ is not achievable in general. The points of X are shown in green and the points of Y are shown in red. The thick red line corresponds to the image of the x-axis through the Eucliden isometry S. See the text for details.*

that the image through S of the x-axis, $S(\{\lambda e_1 | \lambda \in \mathbb{R}\})$, is a line, and that this line must intersect the three balls $\overline{B}(y, \alpha)$, $y \in Y$. It follows that $\alpha \geqslant \sqrt{\varepsilon/2}$, whence the claim.

It is then clear from the reasoning above that, in general, $t = 1$ is not achievable.

We prove in Theorem 1 below that for all compact $X, Y \subset \mathbb{R}^n$, and some constant c_n, $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X,Y) \leqslant c_n \left(d_{\mathcal{GH}}(X,Y) \right)^{1/2}$. To prove this, we resort to the following results of Alestalo et al., [2]. See also [11].

Definition 4. *Let (X, d_X) and (Y, d_Y) be metric spaces. For a map $f : X \to Y$, its* distorsion *is the number*

$$dist(f) := \sup_{x,x' \in X} |d_X(x,x') - d_Y(f(x), f(x'))|.$$

We say that f is an ε-nearisometry if $dist(f) \leqslant \varepsilon$.[6]

Theorem 1 ([2] Thm. 2.2). *Suppose that $X \subset \mathbb{R}^n$ is compact and that $f : X \to \mathbb{R}^n$ is an $\varepsilon \cdot \mathbf{diam}(X)$-nearisometry with $\varepsilon \leqslant 1$. Then there is a Euclidean isometry T s.t. $\|f - T\|_{L^\infty(X)} \leqslant c_n \sqrt{\varepsilon} \cdot \mathbf{diam}(X)$, where c_n depends only on n.*

We then have the main result of this section:

Theorem 2. *Let $X, Y \subset \mathbb{R}^n$ be compact. Then $d_{\mathcal{GH}}(X,Y) \leqslant d_{\mathcal{H},iso}^{\mathbb{R}^n}(X,Y) \leqslant c_n' \cdot M^{\frac{1}{2}} \cdot (d_{\mathcal{GH}}(X,Y))^{\frac{1}{2}}$ where $M = \max(\mathbf{diam}(X), \mathbf{diam}(Y))$ and c_n' is a constant that depends only on n.*

Remark 4. *We can obtain an exponent $t = 1$ by restricting the analysis to the class of compact subsets of $\mathbb{R}^n$ that are suficiently thick, in the sense of [2], details will be provided elsewhere.*

We have therefore proved that when we restrict ourselves to subsets of $\mathbb{R}^n$ endowed with the Euclidean metric, the Gromov-Hausdorff distance and the Euclidean isometries invariant Hausdorff distance are *comparable* or *equivalent*. In the next section we see that the *gap* between these two dissimilarity measures can be closed by modifying expression (5) of the GH distance in a convenient way. This modification will reveal a connection between computing this new GH distance and the EDMCP.

3.2. Closing the gap

We saw in §3.1 that the reason why $d_{\mathcal{GH}}(\Delta_2, \{p\}) = \frac{1}{2}$ instead of $\sqrt{3}/3 = d_{\mathcal{H},iso}^{\mathbb{R}^n}(\Delta_2, \{p\})$ is that the metric $d \in \mathcal{D}(d_X, d_Y)$ minimizing (5) is not Euclidean, but corresponds to embedding both Δ_2 and $\{p\}$ into a space with constant curvature tending to $-\infty$.

In this section we change equation (5) and force the minimization to be performed over "Euclidean" metrics only. We give the precise statements and definitions below.

Definition 5 (Euclidean metrics, [4] §38). *Let (Z, d) be a compact metric space. The metric d is* Euclidean *if and only if there exists $n \in \mathbb{N}$ s.t. (Z, d) can be isometrically embedded into $\mathbb{R}^n$.*[7]

We also say that a metric space is Euclidean when its metric is Euclidean.

When Z is finite, there are simple ways of checking whether a given metric is Euclidean. Below we recall one such characterization that will be useful for our presentation. Let $Z = \{z_1, \ldots, z_\ell\}$ and $D^{(2)}$ be the matrix with elements $d^2(z_i, z_j)$. Let $\mathbb{1}_\ell = (1, 1, \ldots, 1)^T \in \mathbb{R}^\ell$ and $\mathbb{I}_\ell$ be the $\ell \times \ell$ identity matrix. Let $Q_\ell = \mathbb{I}_\ell - \frac{1}{\ell}\mathbb{1}_\ell$. Consider the map $\tau_\ell : \mathbb{R}^{\ell \times \ell} \to \mathbb{R}^{\ell \times \ell}$ given by $A \mapsto -\frac{1}{2}Q_\ell A Q_\ell$.

Proposition 4 ([4] §43). *A necessary and sufficient condition that a semi-metric space (Z, d), $\#Z = \ell$, be isometrically embeddable in some $\mathbb{R}^r$ $(r \in \mathbb{N})$ is that the matrix $\tau_\ell(D^{(2)})$ be positive semidefinite (PSD).*

In the case of a finite Euclidean metric space (Z, d), $Z = \{z_1, \ldots, z_\ell\}$, one says that the matrix $((d(z_i, z_j)))$ is a *Euclidean distance matrix* (EDM).

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[6]Note that we are not requiring that in addition $f(X)$ be an ε-net of Y.

[7]Informally, this means that the metric can be realized by a set of points in some Euclidean space.

Definition 6 (Euclidean Metric Couplings, cf. Definition 3). *For Euclidean $X, Y \in \mathbb{R}^n$ let $\mathcal{D}_\mathcal{E}(d_X, d_Y)$ denote the set of metrics d on $X \sqcup Y$ such that $d(x, x') = \|x - x'\|$, $d(y, y') = \|y - y'\|$ for $x, x' \in X$ and $y, y' \in Y$, and d is* Euclidean.

Remark 5. *Clearly, for Euclidean X, Y, $\mathcal{D}_\mathcal{E}(d_X, d_Y) \neq \varnothing$. Also, notice that if (X, d_X) and (Y, d_Y) are Euclidean, then $\mathcal{D}_\mathcal{E}(d_X, d_Y) \subset \mathcal{D}(d_X, d_Y)$.*

We now proceed to modify the GH distance in order to obtain a related notion of dissimilarity better adapted to Euclidean metric spaces. We propose a modification of (5): Let X and Y be *compact* subsets of $\mathbb{R}^n$ endowed with the Euclidean metric. Consider the following *tentative* distance (cf. (5)):

$$d^\mathcal{E}_{\mathcal{GH}}(X, Y) := \inf_{d \in \mathcal{D}_\mathcal{E}(X,Y)} d^{(X \sqcup Y, d)}_\mathcal{H}(X, Y) \qquad (10)$$

Remark 6. *Notice that solving for the optimal d above can be regarded as an EDMCP, [1, 3]. This family of optimization problems seeks to find an EDM which satisfy certain optimality criteria. Typically, the input is a partial EDM, i.e., a matrix with some missing entries, and the goal is to find an EDM that preserves the entries that are known and, for example, has minimal Frobenius norm. Solutions to these family problems are usualy found via Semidefinite Programming (SDP), [6]. In the case of (10), if $\#X = \ell$ and $\#Y = m$, the goal is to find a matrix $\mathbf{D} \in \mathbb{R}^{\ell \times m}$ with nonnegative elements s.t.*

$$\begin{pmatrix} ((\|x_i - x_j\|)) & \mathbf{D} \\ \mathbf{D}^T & ((\|y_i - y_j\|)) \end{pmatrix}$$

is an EDM and $J(\mathbf{D}) := \max(\max_i \min_j D_{ij}, \max_j \min_i D_{ij})$ is minimized. *In practice, that $\mathbf{D}$ be an EDM is enforced via the condition given by Proposition 4.*

Note that the map $\mathbf{D} \mapsto J(\mathbf{D})$ is non-convex and non-smooth what makes, solving this particular problem, even approximately, very difficult. In contrast, by invoking the L^p Gromov-Hausdorff distances of [17] one can provide more tractable alternatives, see §3.3 below.

Remark 7. *We haven't yet proved that $d^\mathcal{E}_{\mathcal{GH}}(,)$ is a distance, but this will follow from the Theorem below. Also, notice that if (X, d_X) and (Y, d_Y) are Euclidean, by Remark 5 and invoking expression (5), one sees that $d_{\mathcal{GH}}(X, Y) \leqslant d^\mathcal{E}_{\mathcal{GH}}(X, Y)$.*

This is the core result of this section.

Theorem 3. *For $X, Y \subset \mathbb{R}^n$ compact, $d^{\mathbb{R}^n}_{\mathcal{H},iso}(X, Y) = d^\mathcal{E}_{\mathcal{GH}}(X, Y)$.*

Corollary 2. *$d^\mathcal{E}_{\mathcal{GH}}(,)$ is a metric on the set of all isometry classes of compact subsets of $\mathbb{R}^n$.*

3.3. The case of L^p Gromov-Hausdorff distances

The same duality between $d_{\mathcal{GH}}(,)$ and $d^{\mathbb{R}^n}_{\mathcal{H},iso}(,)$ for Euclidean metric spaces is also enjoyed by other notions of distance between shapes (metric spaces) which exhibit a less combinatorial nature. The *counterpart* of $d_\mathcal{H}(,)$ we are alluding to (which we referred to as EWp before) is based on what is known in the Shape/Image Comparison community as Earth Mover's distance (EMD), [22]. This distance is also known as the Wasserstein-Kantorovich-Rubinstein distance and we will denote it by $d_{\mathcal{W}_p}$, where $p \geqslant 1$. We refer the reader to [17] for a discussion of this connection. We give a short account below. To simplify the presentation, we assume that all metric spaces are finite. One assumes that the sets to be compared are specified by *probability measures*, that is, we are given both a subset of $\mathbb{R}^n$ (the *support* of the measure) and a distribution of importance or weights over this subset. This distance takes the following form in the case of *weighted* finite sets (X, μ^X), (Y, μ^Y):[8]

$$d^{\mathbb{R}^n}_{\mathcal{W}_p}(X, Y) := \left(\inf_\mu \sum_{ij} \mu_{ij} \|x_i - y_j\|^p \right)^{1/p} \qquad (11)$$

where μ ranges over the set of *coupling measures* $\mathcal{M}(\mu^X, \mu^Y) := \{\mu \in \mathbb{R}^{\#X \times \#Y} | \mu_{ij} \geqslant 0, \sum_j \mu_{ij} = \mu^X_i, \sum_i \mu_{ij} = \mu^Y_j\}$. These objects can be regarded, as a first approximation, as *fuzzy* correspondences, cf. Definition 2. One can then define $d^{\mathbb{R}^n}_{\mathcal{W}_p,iso}(X, Y)$ in the expected way as:

$$d^{\mathbb{R}^n}_{\mathcal{W}_p,iso}(X, Y) := \inf_{T \in E(n)} d^{\mathbb{R}^n}_{\mathcal{W}_p}(X, T(Y)).$$

Note the similarity of this expression with the ICP objective function, [23].

Now, in this new framework, one must identify the counterpart for $d_{\mathcal{GH}}(,)$, which we referred to as GHp before. The new objects we are dealing with are metric spaces that are also endowed with a weight (probability measure), that is, triples (X, d_X, μ^X). The correct answer is the distance introduced by K.L. Sturm [24, 17] (cf. Definition 5):

$$\mathbf{S}_p((X, d_X, \mu^X), (Y, d_Y, \mu^Y)) := \left(\min_{d,\mu} \sum_{ij} \mu_{ij} d^p_{ij} \right)^{1/p} \qquad (12)$$

where $\mu \in \mathcal{M}(\mu^X, \mu^Y)$ and $d \in \mathcal{D}(d_X, d_Y)$. When X and Y are Euclidean, one can now define $\mathbf{S}^\mathcal{E}_p(X, Y)$ by minimizing over $d \in \mathcal{D}_\mathcal{E}$ in (12). The main two results

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[8]This means that μ^X is a collection of $\#X$ non-negative real numbers whose sum is 1.

we want to report are, basically, the respective versions of Theorems 2 and 3.

Let $M := \max(\mathbf{diam}\,(X), \mathbf{diam}\,(Y))$. It is easy to check that $\mathbf{S}_p(X,Y) \leqslant M$ for all $p \geqslant 1$, [17].

The proof of Theorem 4 below relies on the following *key* Lemma, of independent interest:[9]

Lemma 2. *Assume X and Y are finite weighted metric spaces with $\mathbf{S}_p((X, d_X, \mu^X), (Y, d_Y, \mu^Y)) \leqslant \varepsilon \cdot M$, and $\varepsilon \leqslant 1$. Then for all $\gamma > 0$ and $\alpha, \beta \geqslant 0$ s.t. $\alpha + \beta = 1$, there exist $X_\varepsilon \subset X$, $Y_\varepsilon \subset Y$, $R \in \mathcal{R}(X_\varepsilon, Y_\varepsilon)$ and $\mu \in \mathcal{M}(\mu^X, \mu^Y)$ s.t. $\min\left(\mu^X(X_\varepsilon), \mu^Y(Y_\varepsilon), \mu(R)\right) \geqslant 1 - \varepsilon^{p\beta}/\gamma^p$ and $d_{\mathcal{GH}}(X_\varepsilon, Y_\varepsilon) \leqslant \gamma \varepsilon^\alpha \cdot M$.*

Remark 8. *Note that the preceeding Lemma says that if $\mathbf{S}_p(X,Y)$ is small, then one can find a part of X and a part of Y which can be put in correspondence, and s.t. their GH distance is also small. Moreover, these parts have large total weight. Also, with small values of p, the $\mathbf{S}_p$ distance tolerates large distorsions in areas of small weight, whereas as $p \uparrow \infty$, the behaviour is similar to that of $d_{\mathcal{GH}}(,)$. This is exactly what is desired from* Partial Shape Matching *techniques. A converse of this Lemma also holds. Details will be presented elsewhere.*

Theorem 4. *Let $X, Y \subset \mathbb{R}^n$ be compact and $p \geqslant 1$, then*
$$\mathbf{S}_p(X,Y) \leqslant d^{\mathbb{R}^n}_{\mathcal{W}_p, iso}(X,Y) \leqslant M^{3/4} \cdot c''_n \cdot (\mathbf{S}_p(X,Y))^{\frac{1}{4}}$$
where c''_n is a constant only dependent on n.

Proof. We only prove the rightmost inequality. Let $1 \geqslant \varepsilon = \mathbf{S}_p(X,Y)/M$. Apply the preceding Lemma with $\alpha = \beta = 1/2$ and $\gamma = 1$. Let $X_\varepsilon, Y_\varepsilon, R$ and μ be given by that Lemma. Then, one has $d_{\mathcal{GH}}(X_\varepsilon, Y_\varepsilon) \leqslant \varepsilon^{1/2} \cdot M$.

By Proposition 1 item 3, this implies that there exist a map $\phi : X_\varepsilon \to Y_\varepsilon$ with $\mathrm{dist}(\phi) \leqslant 2\varepsilon^{1/2} M$ and such that $\phi(X_\varepsilon)$ is a $2\varepsilon^{1/2} M$ net for Y_ε. By invoking Theorem 1 we get $T \in E(n)$ s.t. $\max_{x \in X_\varepsilon} \|T(x) - \phi(x)\| \leqslant \varepsilon^{1/2} c_n \sqrt{2} M$. We will use the inequality $(a + b)^p \leqslant 2^{p-1}(a^p + b^p)$ (valid for all $a, b \geqslant 0$) in the proof without further mention.

Now, consider $\sum_{x \in X, y \in Y} \|T(x) - y\|^p \mu_{xy} =$

$$\underbrace{\sum_{(x,y) \in R} \|T(x) - y\|^p \mu_{xy}}_{(\mathbf{I})} + \underbrace{\sum_{(x,y) \in R^c} \|T(x) - y\|^p \mu_{xy}}_{(\mathbf{II})}.$$

Note that $(\mathbf{II})$ can be bounded easily as follows: $(\mathbf{II}) \leqslant \varepsilon^{p/2} 2^p L^p$ where $L := \max(\|x\|_{L^\infty(X)}, \|y\|_{L^\infty(Y)})$. Note that w.l.o.g. we can assume that the circumcenters of X and Y coincide with the origin of $\mathbb{R}^n$, hence we have $L \leqslant M$.

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[9]For simplicity, we assume X and Y to be finite, the results hold however, without this restriction.

It takes a bit more work to bound $(\mathbf{I})$. We have: $\|T(x) - y\|^p \leqslant 2^{p-1}(\|T(x) - \phi(x)\|^p + \|\phi(x) - y\|^p)$ and hence

$$\frac{(\mathbf{I})}{2^{1-p}} \leqslant \underbrace{\sum_{(x,y) \in R} \|T(x) - \phi(x)\|^p \mu_{xy}}_{(\mathbf{A})} + \underbrace{\sum_{(x,y) \in R} \|\phi(x) - y\|^p \mu_{xy}}_{(\mathbf{B})}.$$

We bound $(\mathbf{A}) \leqslant \max_{x \in X_\varepsilon} \|T(x) - \phi(x)\|^p \sum_{(x,y) \in R} \mu_{xy} \leqslant \varepsilon^{p/2} \cdot c_n^p M^p 2^{p/2}$ since $\mu(R) \leqslant 1$. And also, Similarly, we bound $(\mathbf{B}) \leqslant \max_{(x,y) \in R} \|\phi(x) - y\|^p \leqslant \varepsilon^{p/2} \cdot 2^{p/2} M^p$ where the last inequality follows from the fact that $\phi(X_\varepsilon)$ is a $2\varepsilon^{1/2}$-net for Y_ε.

Now, after putting it all together and performing algebraic manipulations (using the fact that $\varepsilon \leqslant 1$) we obtain $\sum_{x \in X, y \in Y} \|T(x) - y\|^p \mu_{xy} \leqslant \varepsilon^{1/4} \cdot M \cdot 8 \max(1 + c_n/\sqrt{2})$. This concludes the proof. $\square$

We also have (cf. Theorem 3):

Theorem 5. *For $X, Y \subset \mathbb{R}^n$ compact and $p \geqslant 1$, $\mathbf{S}_p^{\mathcal{E}}(X,Y) = d^{\mathbb{R}^n}_{\mathcal{W}_p, iso}(X,Y)$.*

We will omit the proof of Theorem 5 since it is rather technical. This theorem offers an interesting alternative to computing $d^{\mathbb{R}^n}_{\mathcal{W}_p, iso}(,)$.

Notice that in the statement of the condition for checking that a given distance matrix is Euclidean (Proposition 4) there appear only *linear functions of squared distances*. Therefore, it is of special interest to consider the possibility of computing

$$\mathbf{S}_2^{\mathcal{E}}(X,Y) := \left(\min_{d,\mu} \sum_{x,y} d^2(x,y)\mu(x,y) \right)^{1/2} \tag{13}$$

where $d \in \mathcal{D}_{\mathcal{E}}(d_X, d_Y)$ and $\mu \in \mathcal{M}(\mu_X, \mu_Y)$. Obviously, for Euclidean X and Y, $\mathbf{S}_2(X,Y) \leqslant \mathbf{S}_2^{\mathcal{E}}(X,Y)$.

Remark 9. *Note that the set of constraints for d in (13) is substantially easier to deal with than in the general setting ($d \in \mathcal{D}(d_X, d_Y)$). Then, we observe that the optimization problem in (13) despite being of bilinear nature, can be efficiently implemented by invoking Proposition 4. Indeed, assume $\#X = \ell$, $\#Y = m$ and that μ^X and μ^Y are the uniform probability measures. Recall the definition of $Q_{\ell+m}$ in the discussion after Definition 5. The optimization problem one needs to solve in practice is: $\min \sum_{ij} P_{ij} U_{ij}$ where $P, U \in \mathbb{R}_+^{\ell \times m}$, $U \mathbb{1}_m = \frac{1}{\ell} \mathbb{1}_\ell$, $U^T \mathbb{1}_\ell = \frac{1}{m} \mathbb{1}_m$ and*

$$-\frac{1}{2} Q_{\ell+m} \begin{pmatrix} ((d_X^2)) & P \\ P^T & ((d_Y^2)) \end{pmatrix} Q_{\ell+m} \succcurlyeq 0.^{[10]}$$

This optimization problem can be attacked using alternate projections method and the well known trick for

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[10]For a matrix A, $A \succcurlyeq 0$ means that A is PSD.

projecting onto the space of PSD matrices (see [6]). Details about our implementation will be presented elsewhere.

4. Conclusions

We have established that the GH and EH distances are comparable for Euclidean metric spaces. We also established comparability in the context of the L^p-GH distances, namely that GHp and EWp are comparable in the class of Euclidean metric spaces. We have shown how the EH distance can be recast as a constrained GH distance by a certain modification of (5). This equivalent expression for EH exposes the connection with the family of EDMCPs. This translation can also be carried out in the context of GHp distances. Using this connection, we design a new procedure for matching point sets under Euclidean invariances. Some auxiliry results of independent interest are presented along the way. We believe that the results in this paper increase the understanding of the methods available for shape matching/comparison.

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