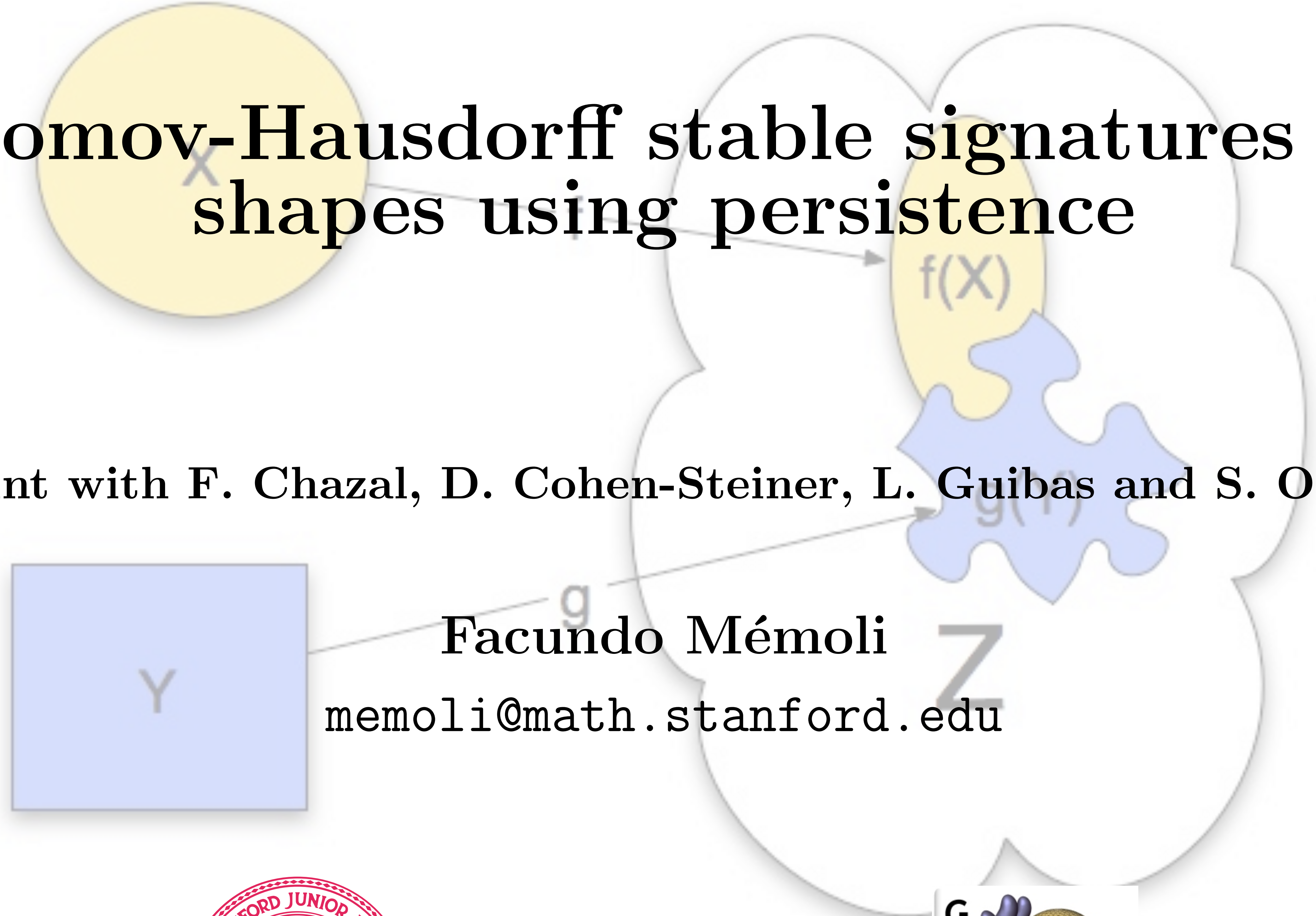


Gromov-Hausdorff stable signatures for shapes using persistence

joint with F. Chazal, D. Cohen-Steiner, L. Guibas and S. Oudot

Facundo Mémoli

`memoli@math.stanford.edu`

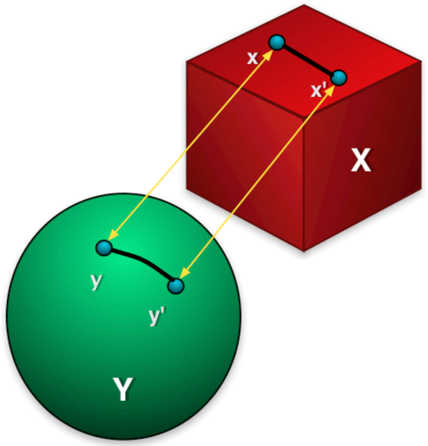
Goal

- Shape discrimination is a very important problem in several fields.

- **Isometry invariant** shape discrimination has been approached with different tools, mostly via computation and comparison of **invariant signatures**, [**HK03,Osada-02,Fro90,SC-00**].

- The **Gromov-Hausdorff distance** (and certain variants) provides a **rigorous** and well motivated framework for studying shape matching under invariances [**MS04,MS05,M07,M08**].

- However, its direct computation leads to **NP hard** problems (BQAP: bottleneck quadratic assignment problems).

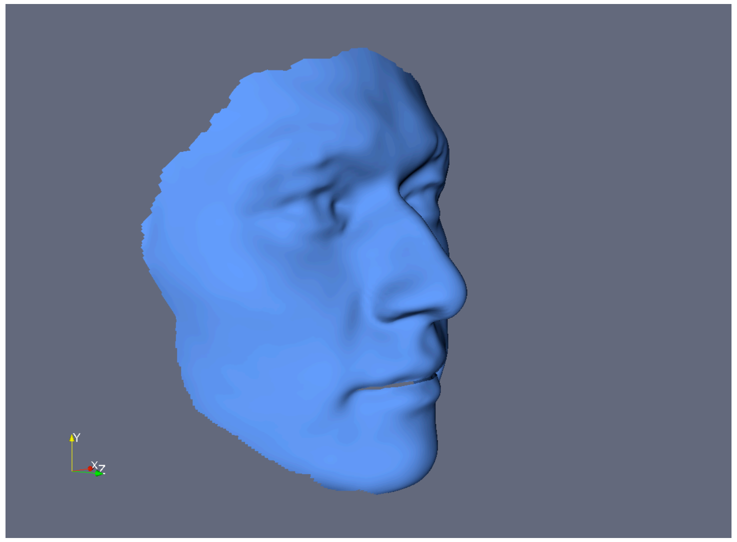


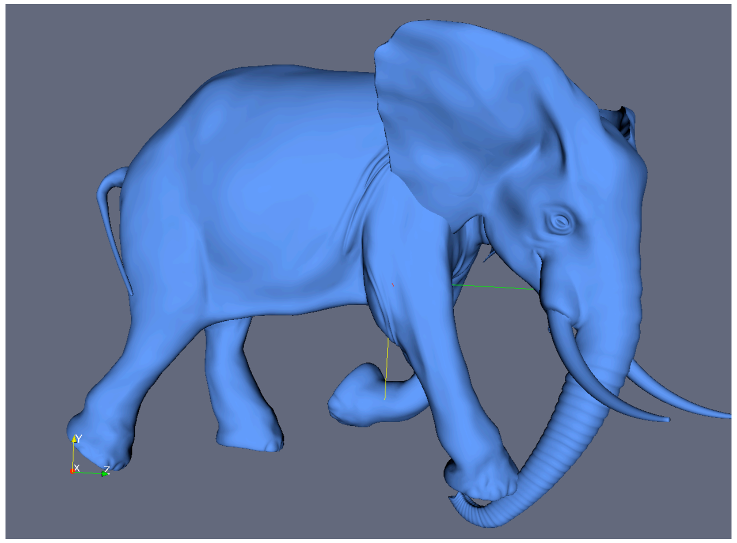
- Most of the effort has gone into finding **lower bounds** for the GH distance that use informative invariant signatures and lead to easier optimization problems [**M07,M08**].

- Using **persistent topology** [**ELZ00**], we obtain a new family of signatures and prove that they are **stable** w.r.t the GH distance: i.e., **we obtain lower bounds for the GH distance!**

- These lower bounds:
  - perform very well in practical application of shape discrimination.
  - lead to **BAP**s (bottleneck assignment problems) which can be solved in **polynomial time**.

visual summary





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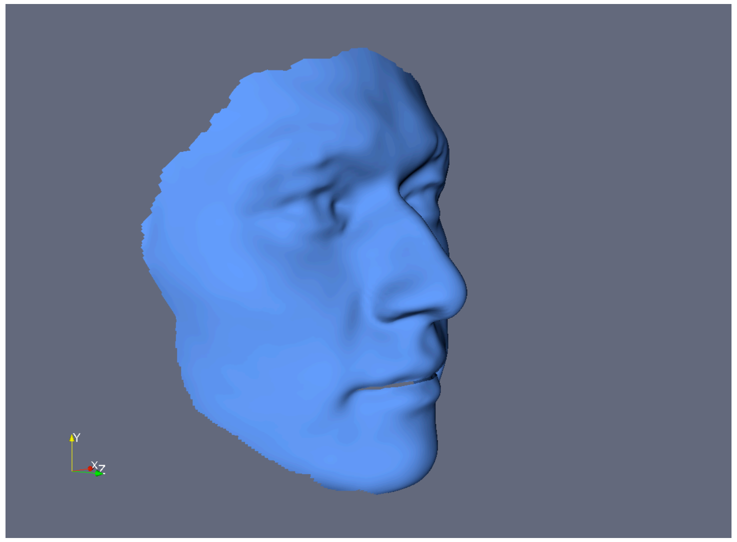

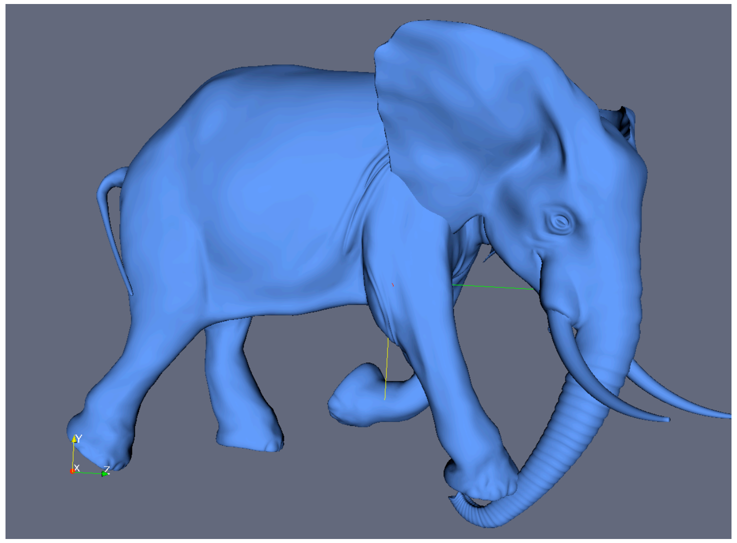

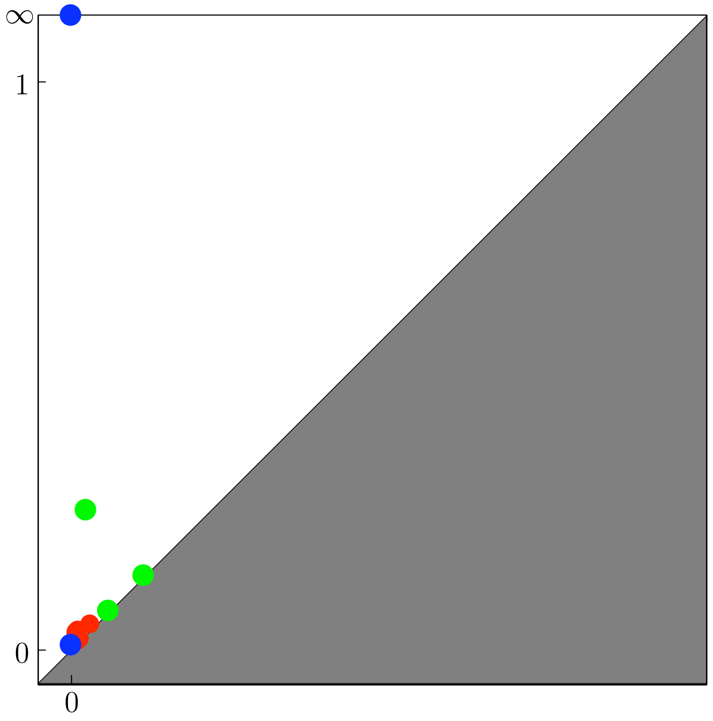

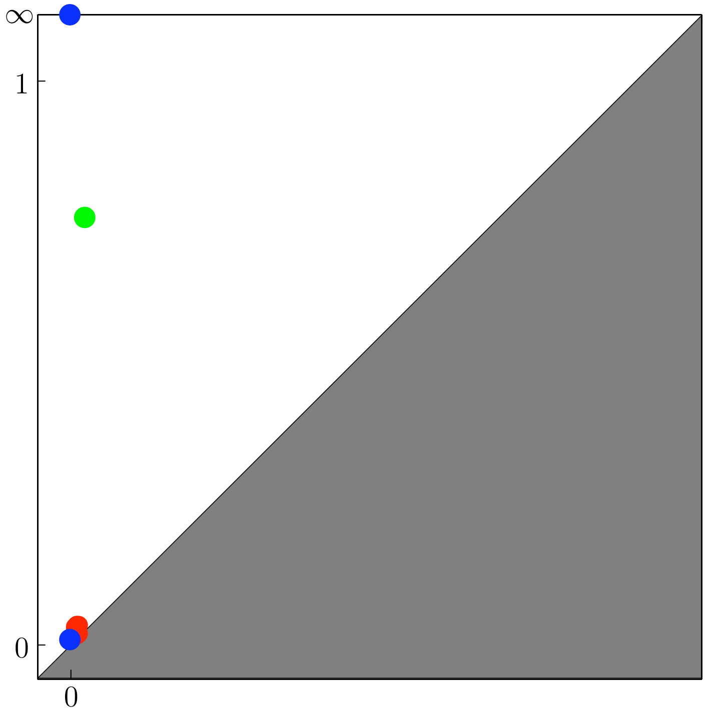

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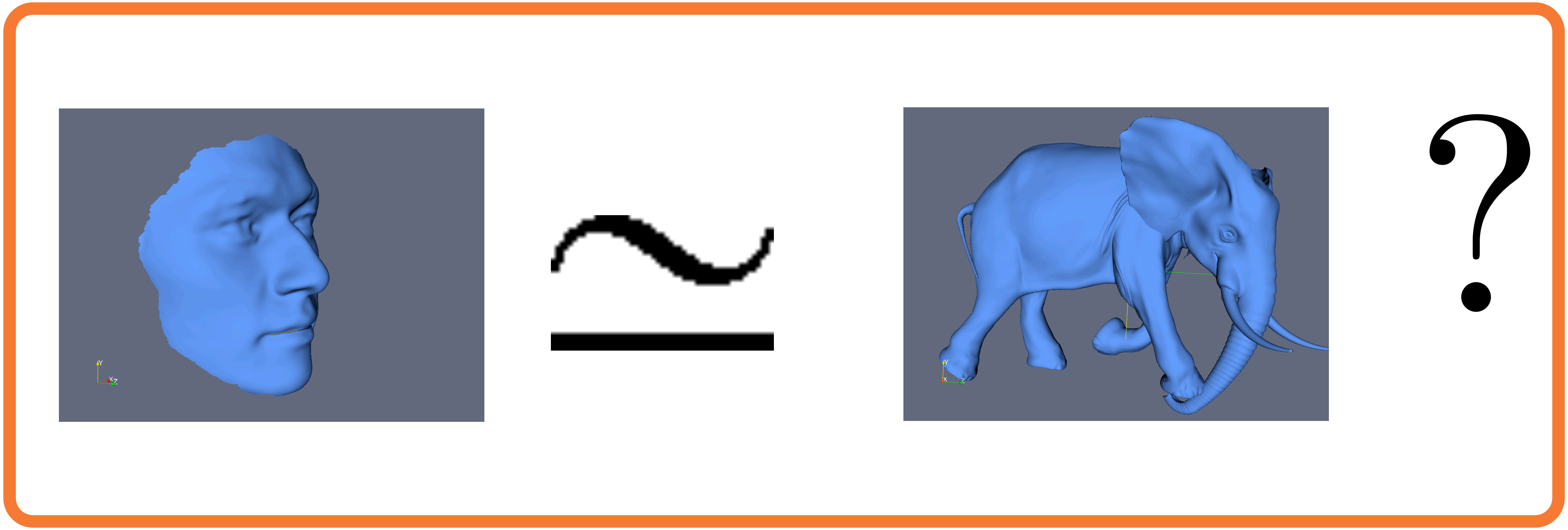

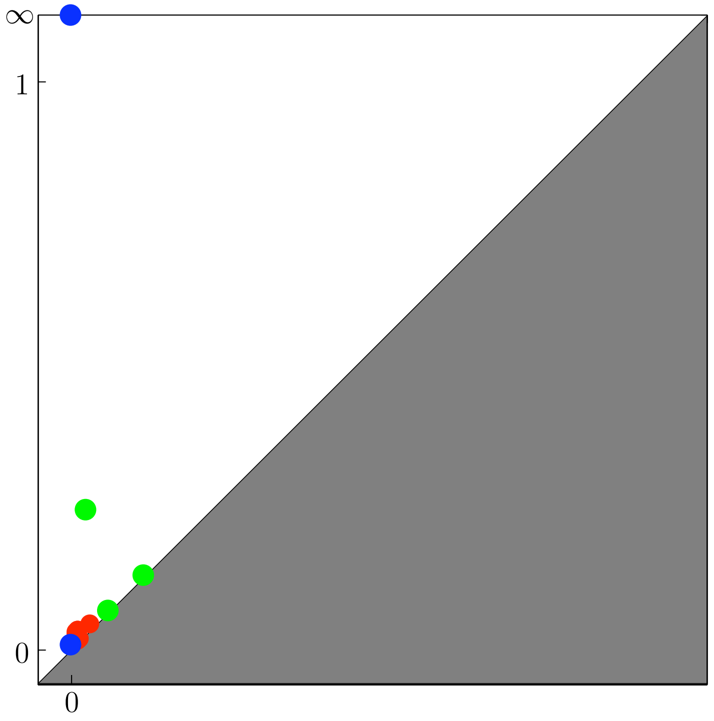

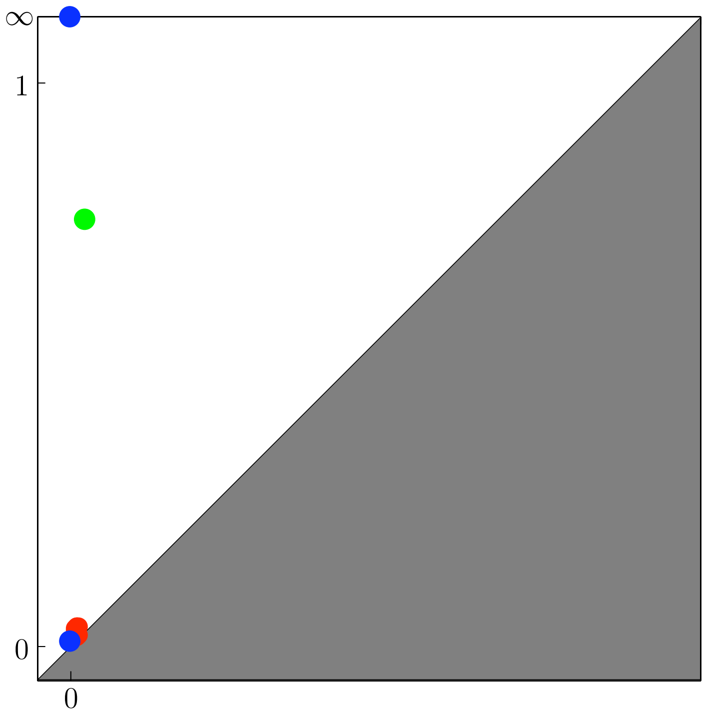

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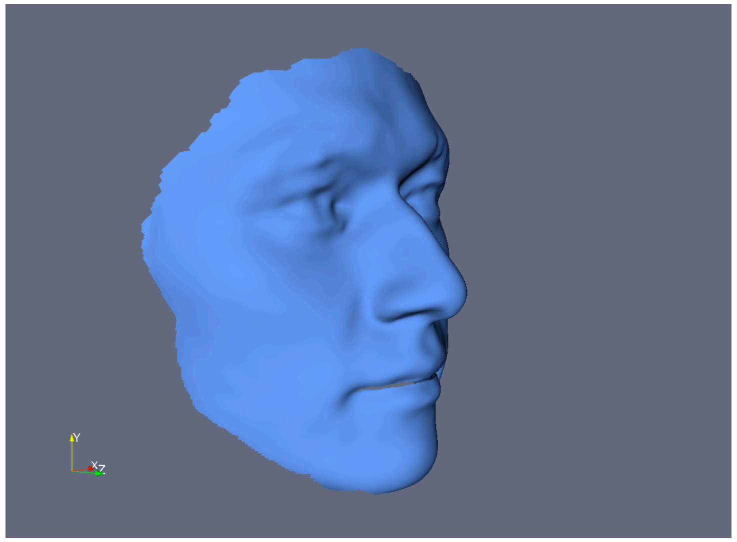
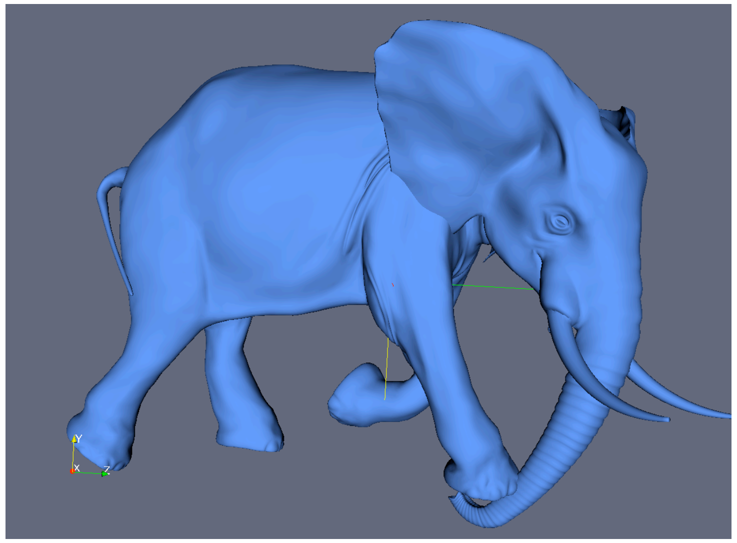
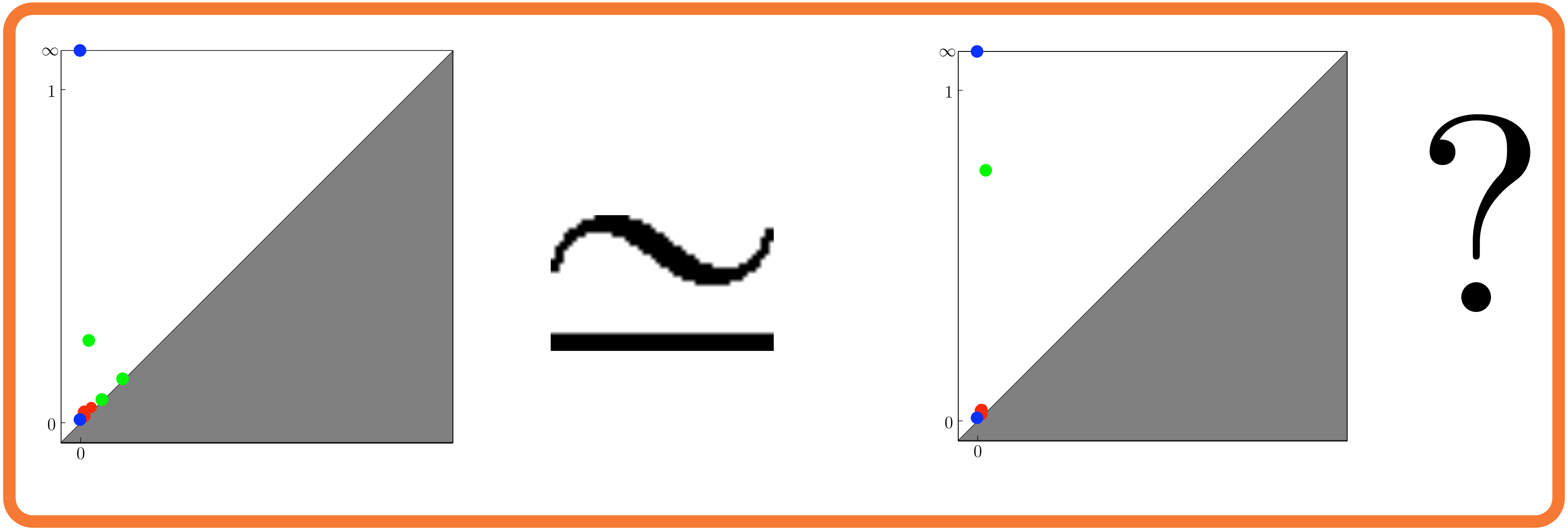
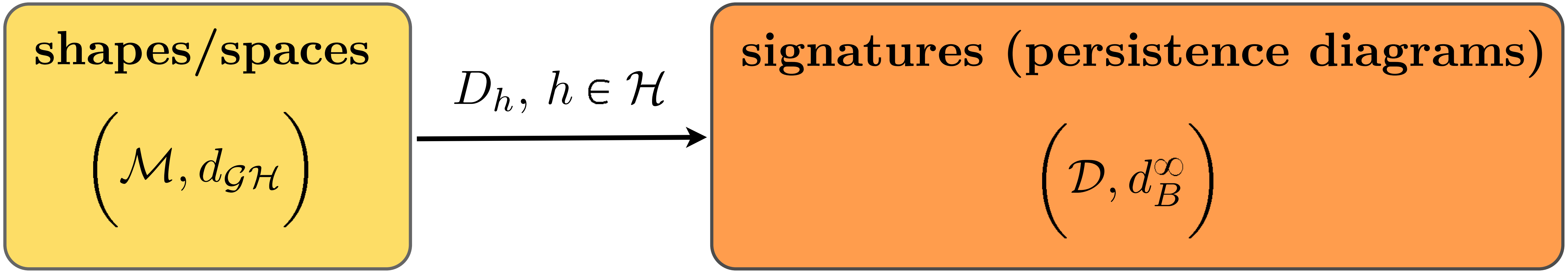






- $\mathcal{M}$: collection of all shapes (finite metric spaces).
- $\mathcal{D}$: collection of all signatures (persistence diagrams).



shapes/spaces

$$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$$

$D_h,\ h \in \mathcal{H}$

signatures (persistence diagrams)

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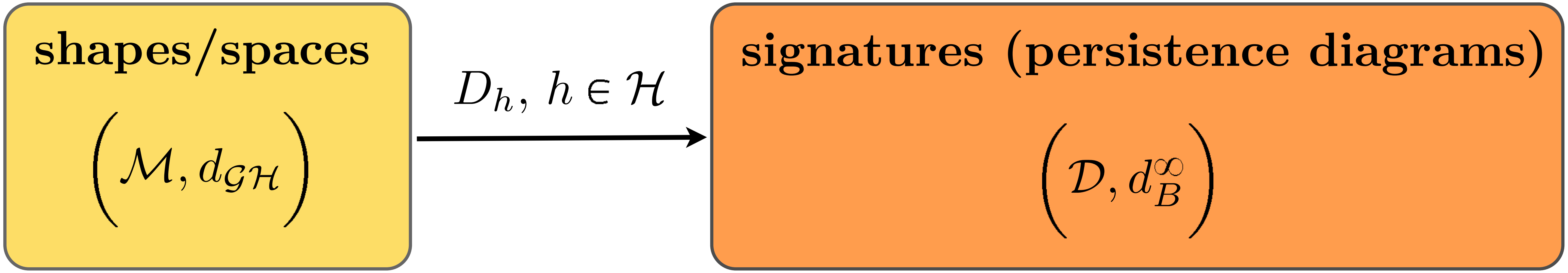

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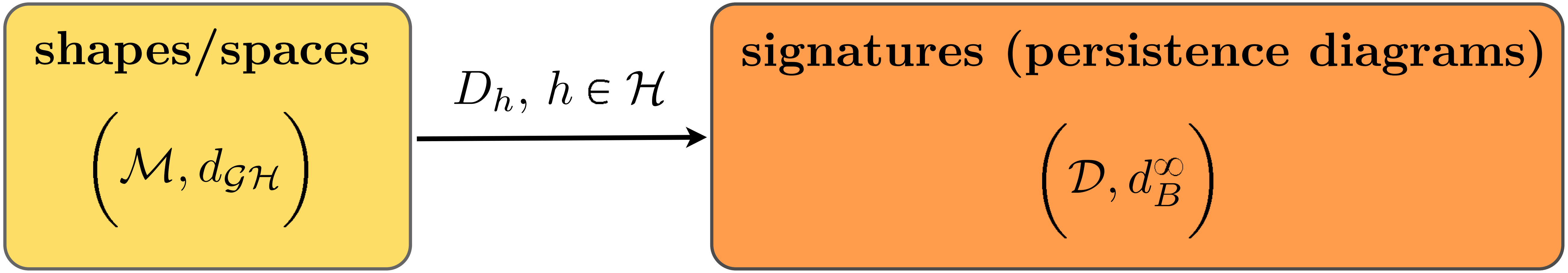

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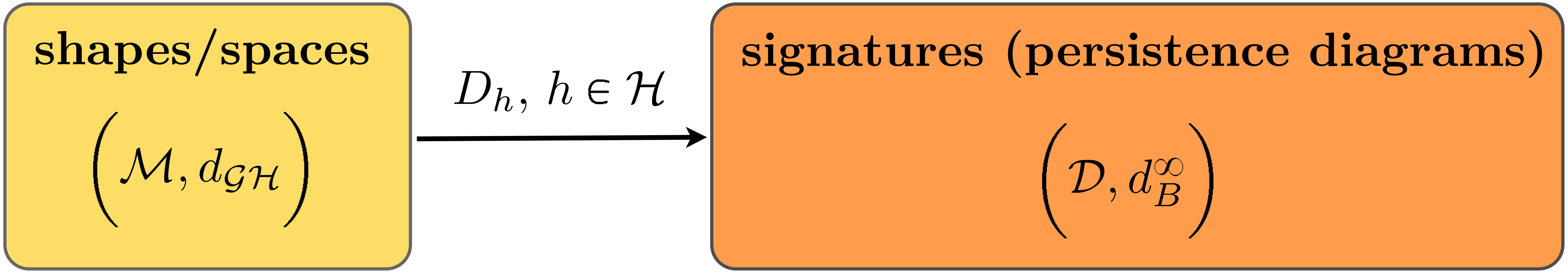

shapes/spaces

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$$\left(\mathcal{D}, d_B^\infty\right)$$

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$$d_{\mathcal{GH}}(X, Y)$$

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shapes/spaces

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$$\xrightarrow{D_h,\ h \in \mathcal{H}}$$

signatures (persistence diagrams)

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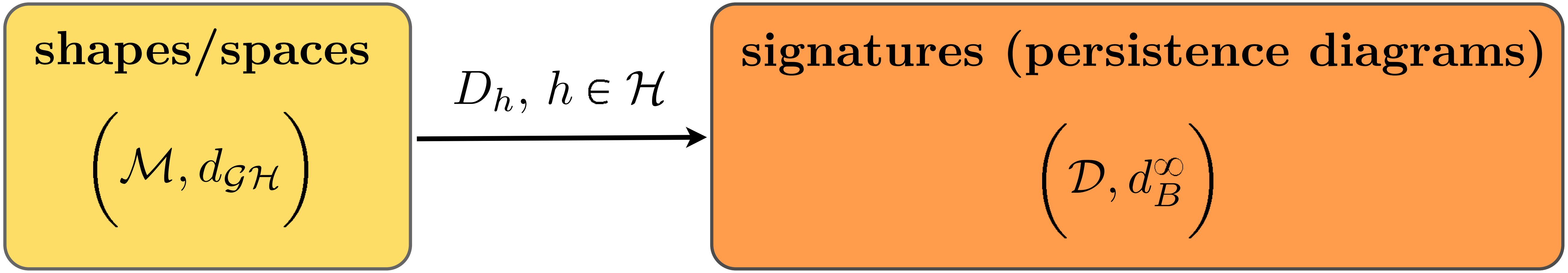

shapes/spaces

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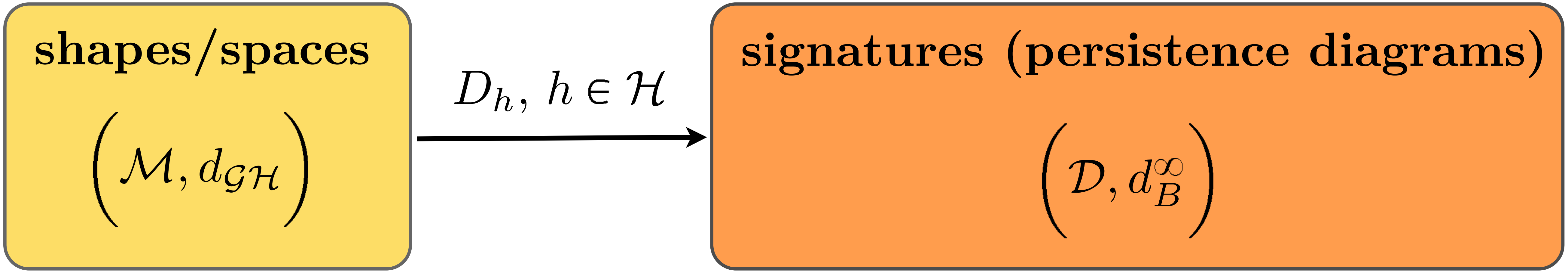


shapes/spaces

$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$

$D_h,\ h \in \mathcal{H}$

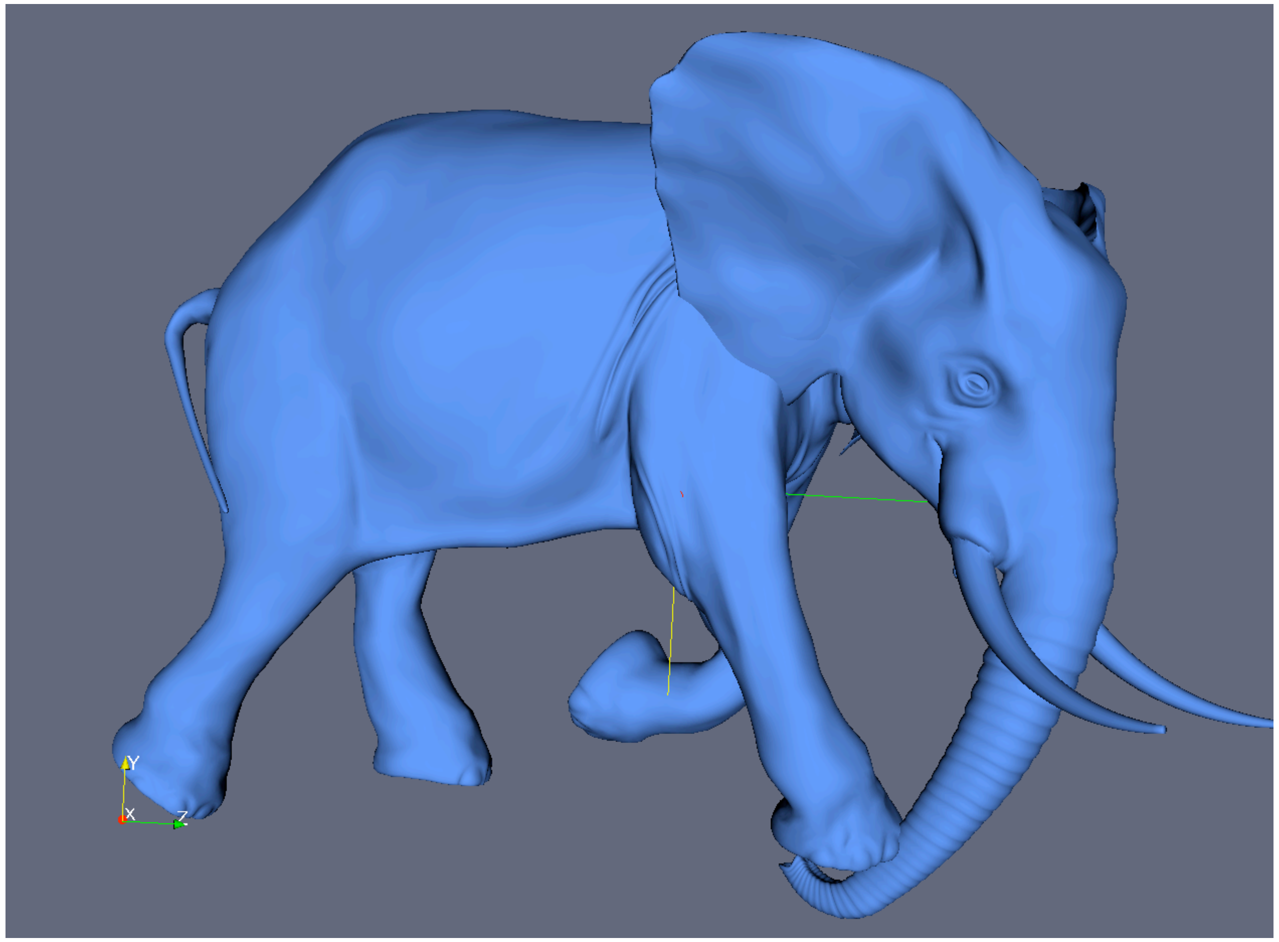
signatures (persistence diagrams)

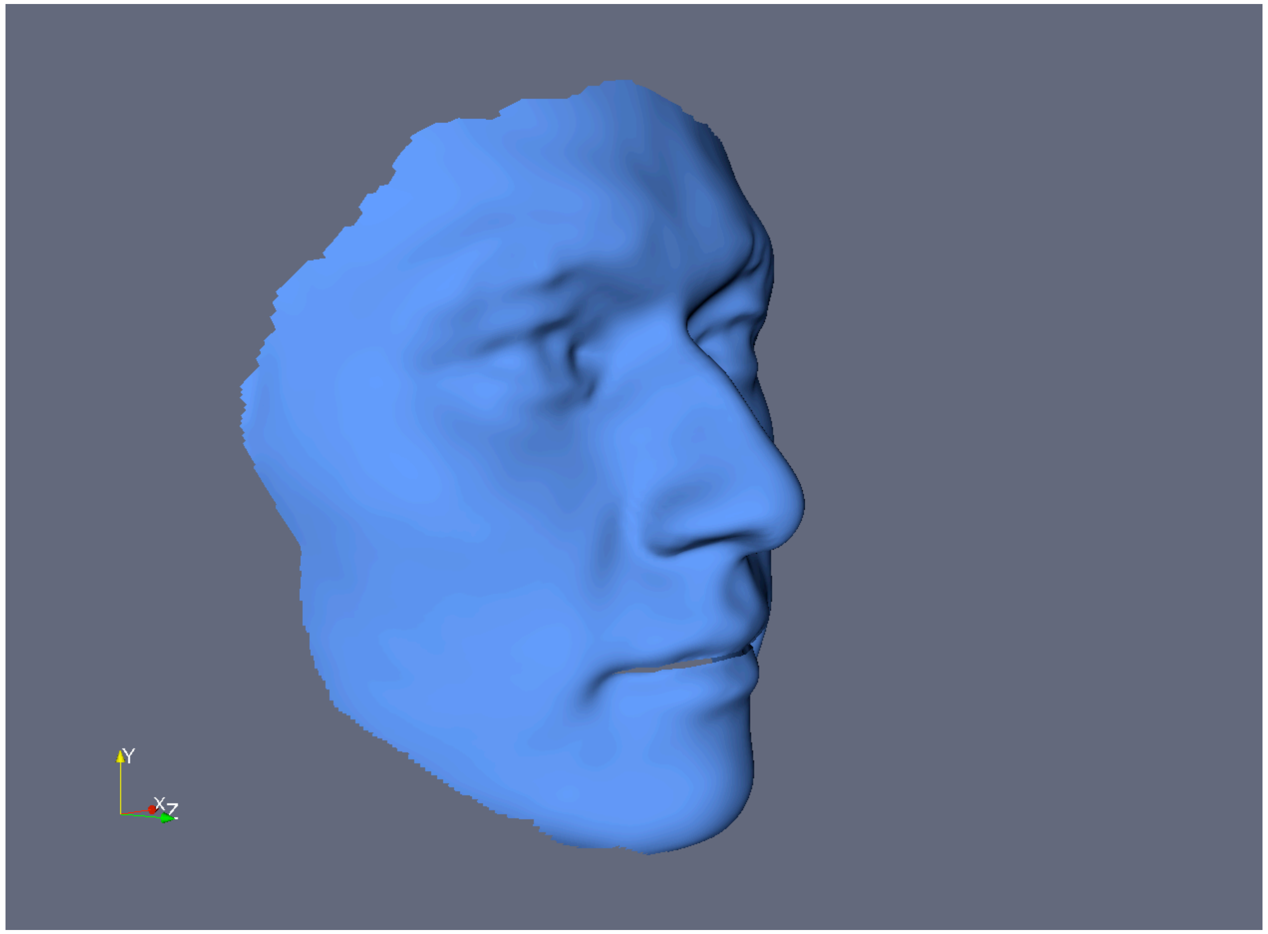
$\left(\mathcal{D}, d_B^{\infty}\right)$

X, Y $\quad$ $D_h(X),\ D_h(Y)$ $\quad$ $h \in \mathcal{H}$

$$d_{\mathcal{GH}}(X, Y) \quad \geqslant \quad d_B^{\infty}(D_h(X),\ D_h(Y))$$

Shapes as metric spaces





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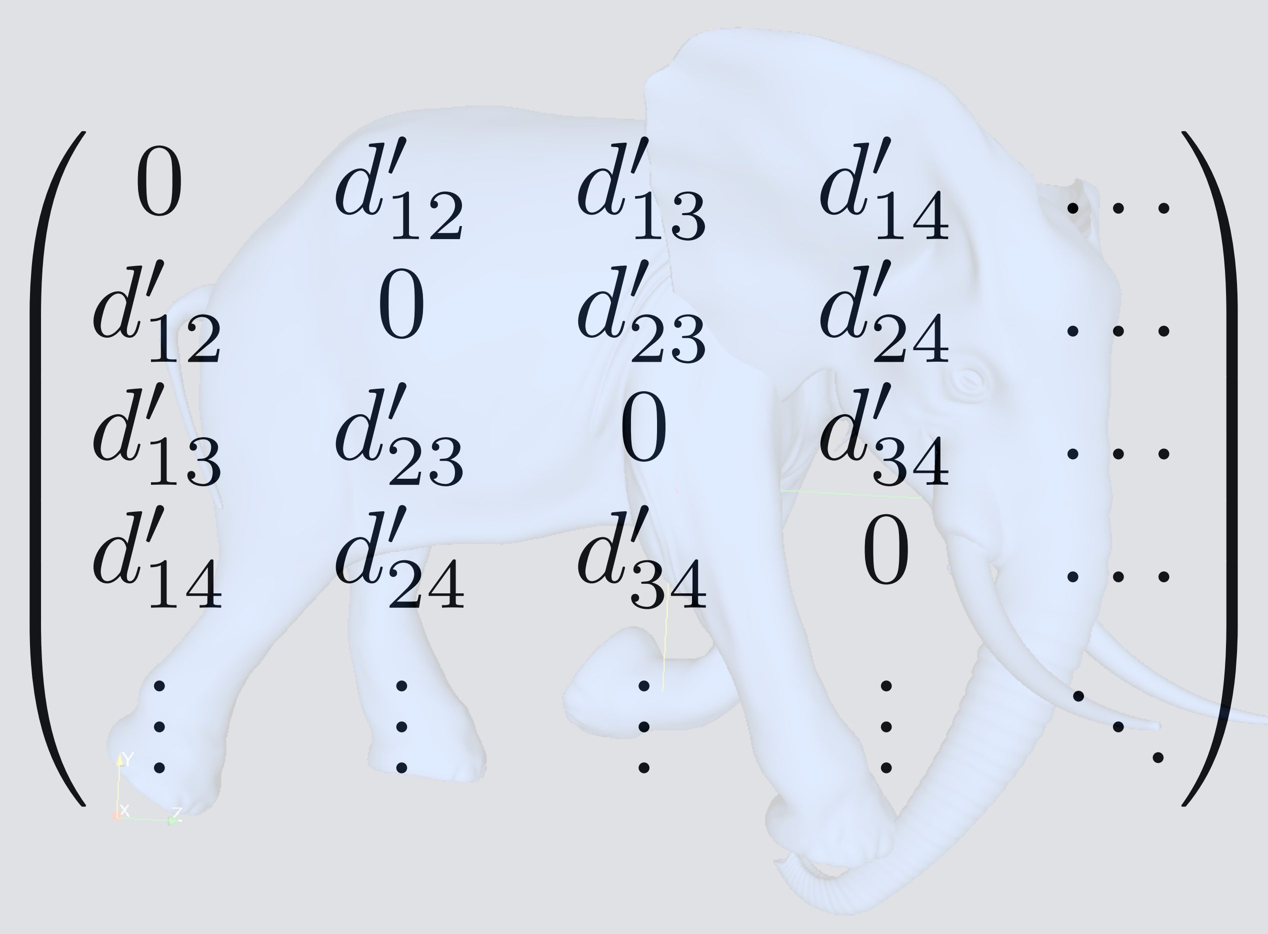
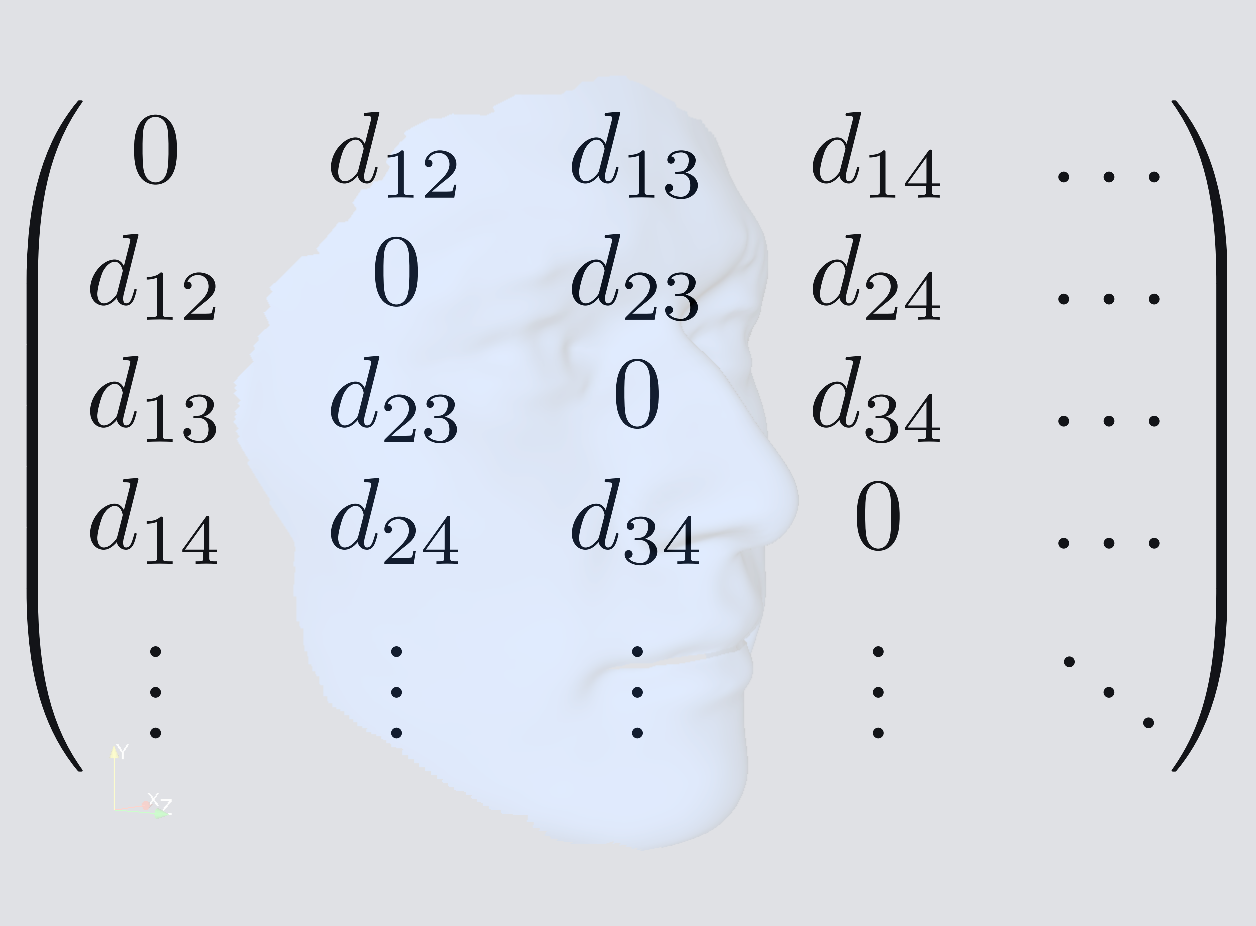


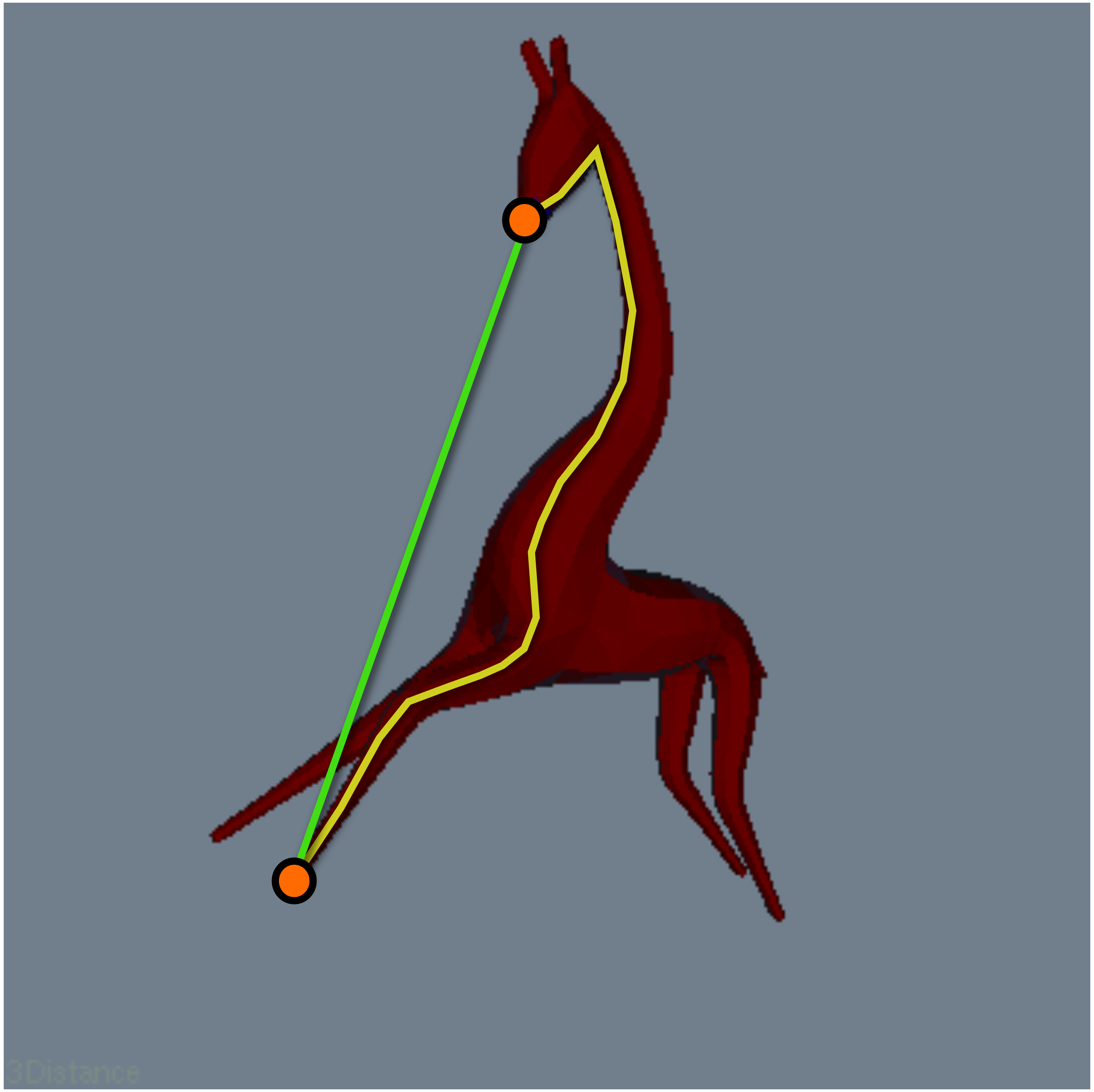
$$\begin{pmatrix} 0 & d'_{12} & d'_{13} & d'_{14} & \cdots \\ d'_{12} & 0 & d'_{23} & d'_{24} & \cdots \\ d'_{13} & d'_{23} & 0 & d'_{34} & \cdots \\ d'_{14} & d'_{24} & d'_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



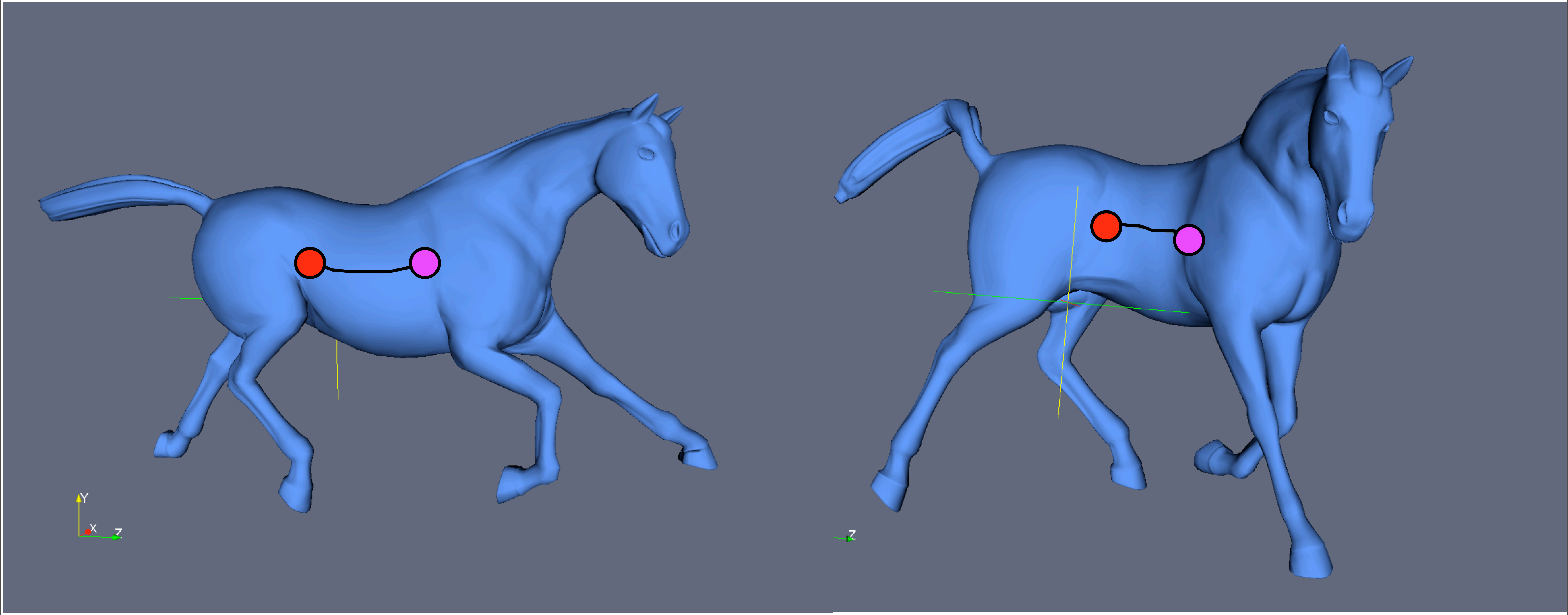
$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then use Gromov-Hausdorff distance..

Choice of the metric: geodesic vs Euclidean



Invariance to isometric deformations (change in pose)



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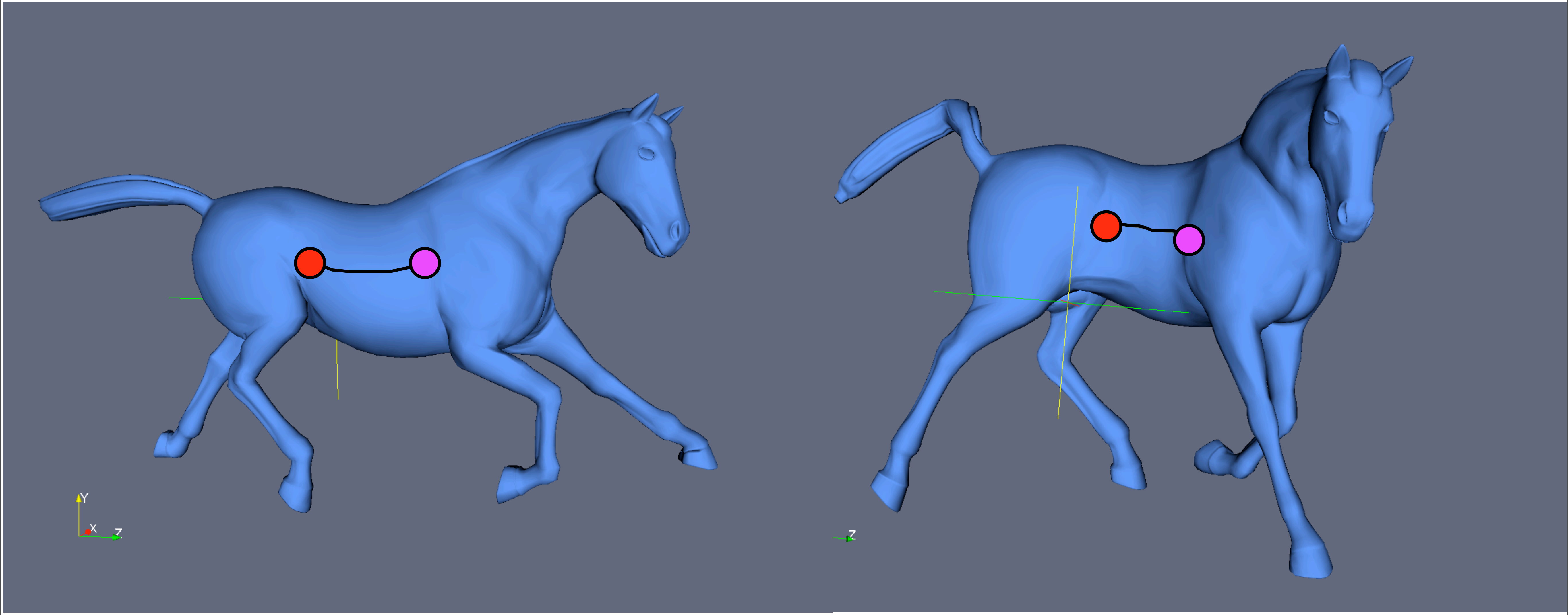
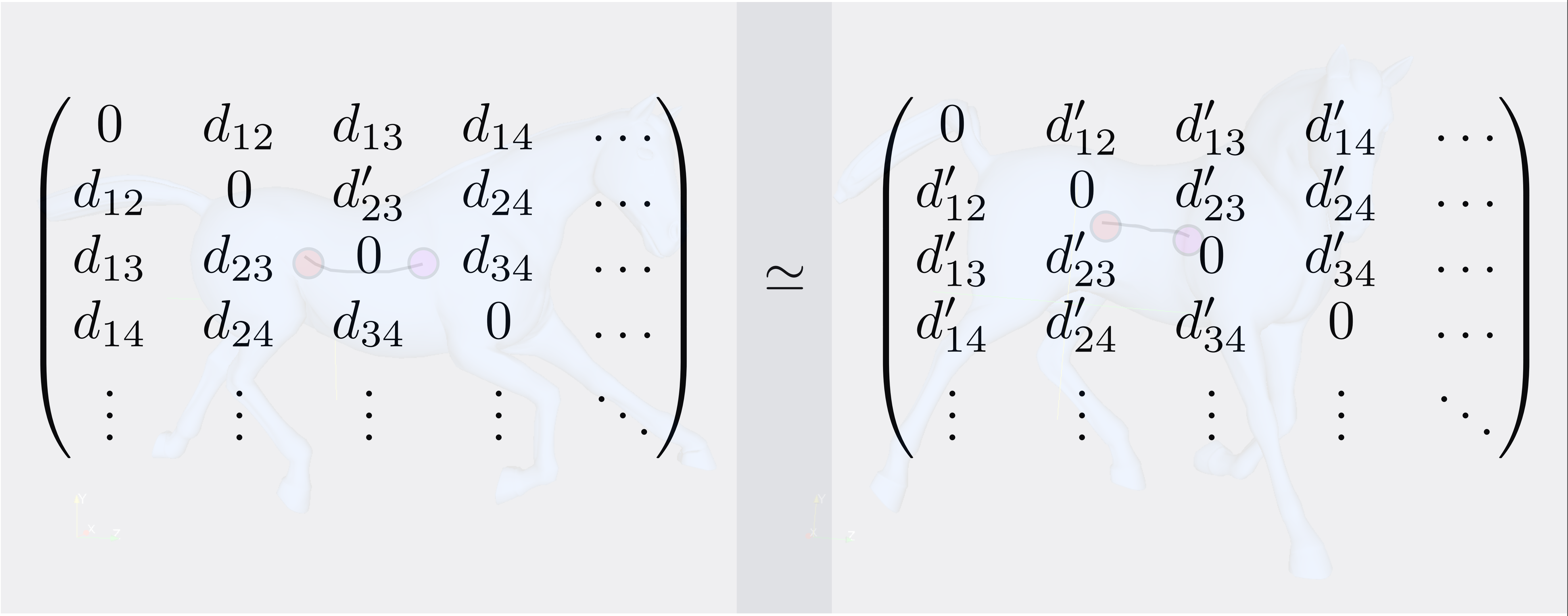


geodesic distance remains approximately constant

Invariance to isometric deformations (change in pose)



$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d'_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \simeq \begin{pmatrix} 0 & d'_{12} & d'_{13} & d'_{14} & \cdots \\ d'_{12} & 0 & d'_{23} & d'_{24} & \cdots \\ d'_{13} & d'_{23} & 0 & d'_{34} & \cdots \\ d'_{14} & d'_{24} & d'_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

geodesic distance remains approximately constant

Definition [Correspondences]

For finite sets A and B, a subset $C \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\ a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\ b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{C}(A, B)$ denote all possible correspondences between sets A and B.

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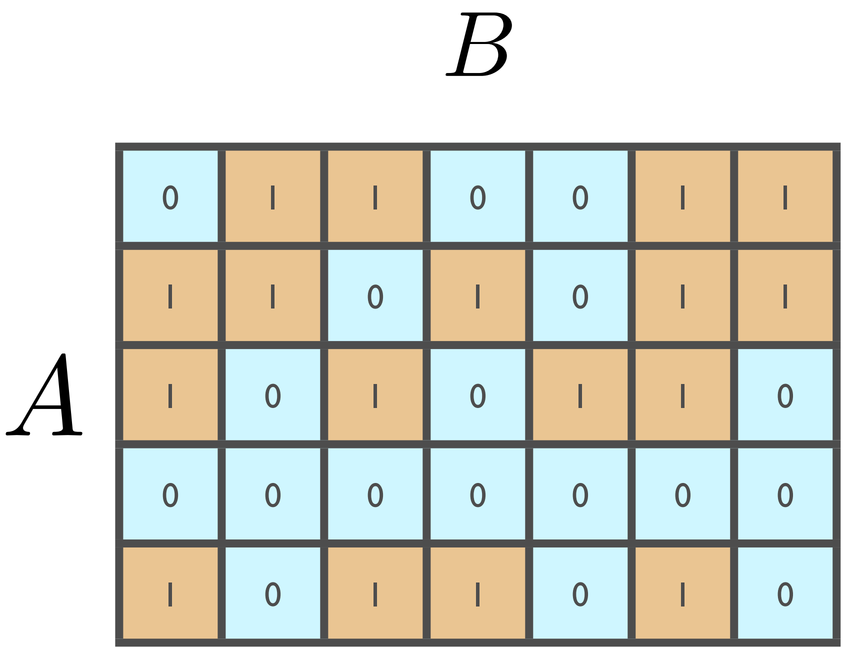

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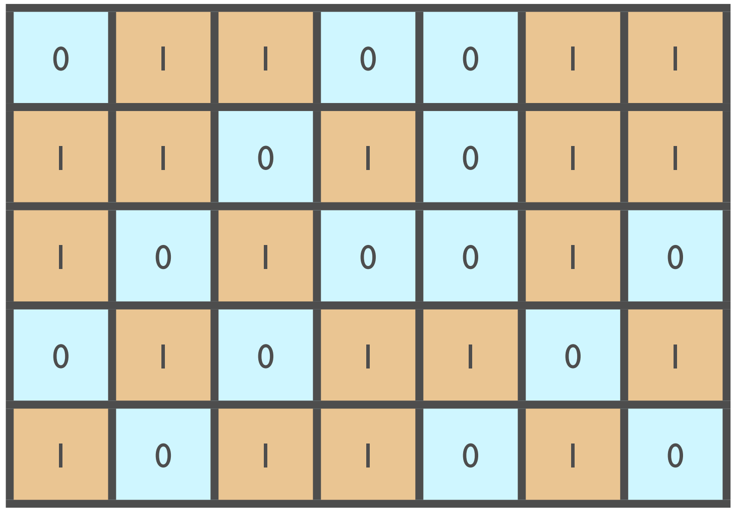

0	1	1	0	0	1	1
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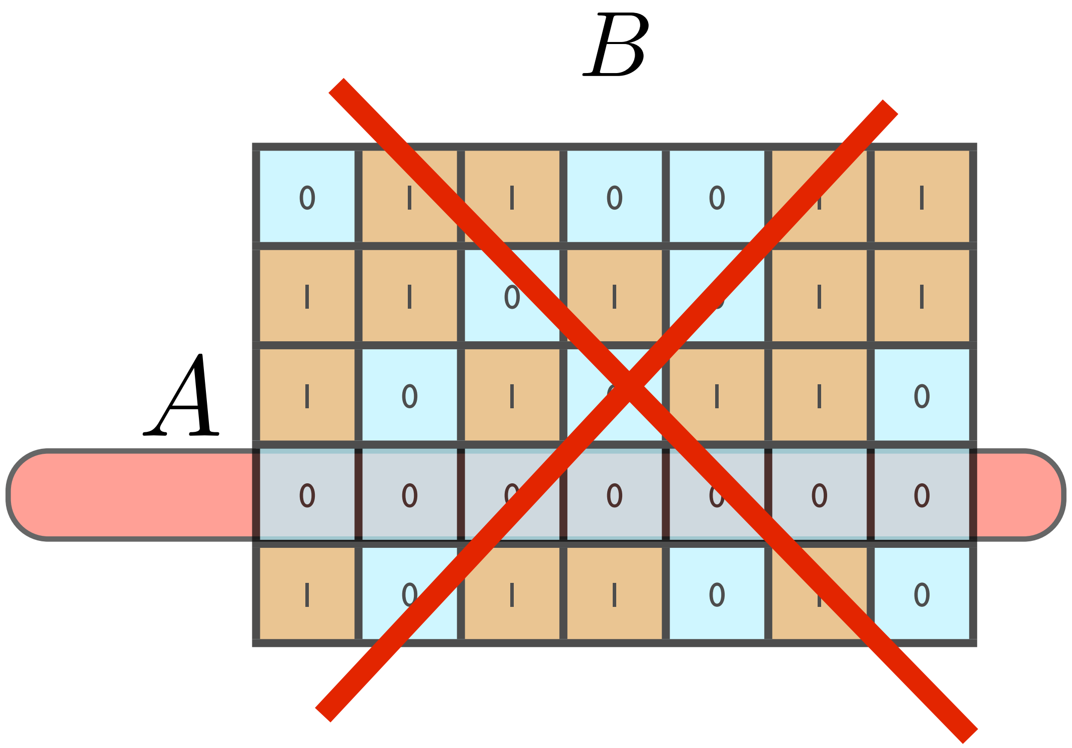

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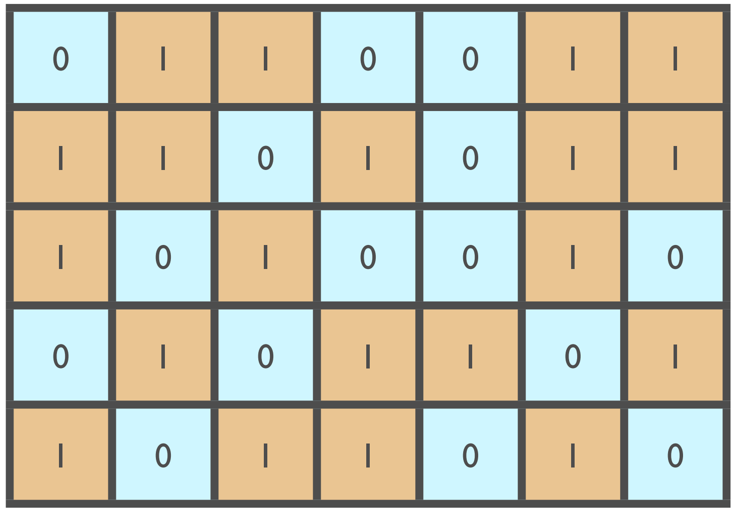

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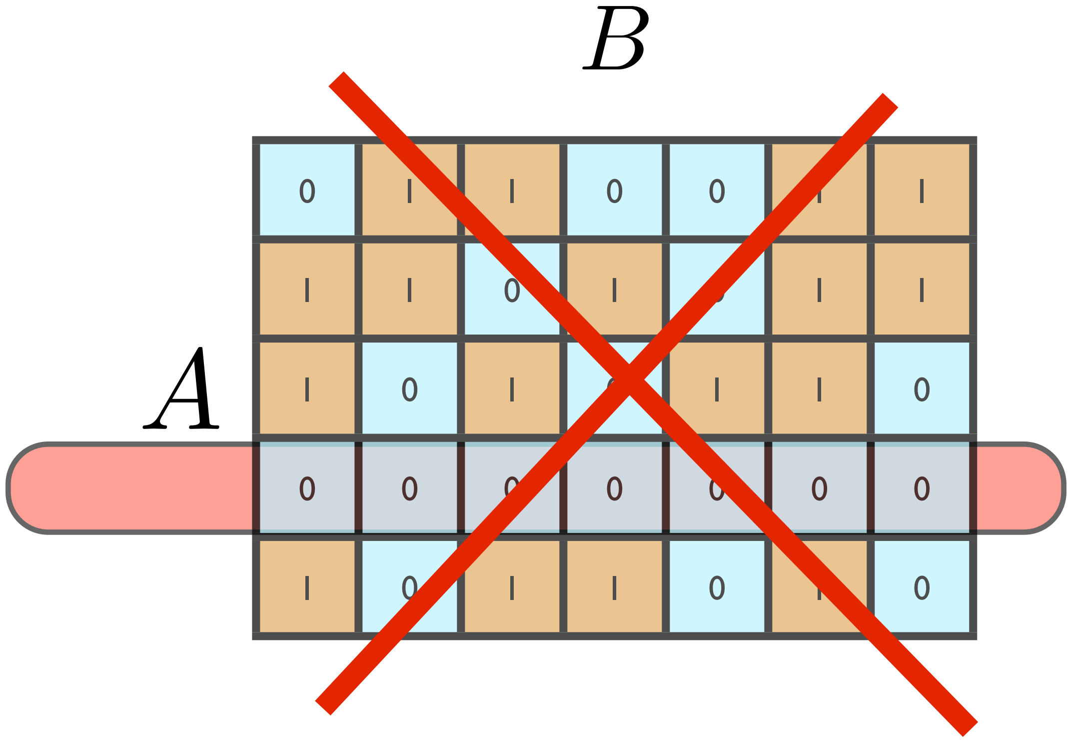

0	1	1	0	0	1	1
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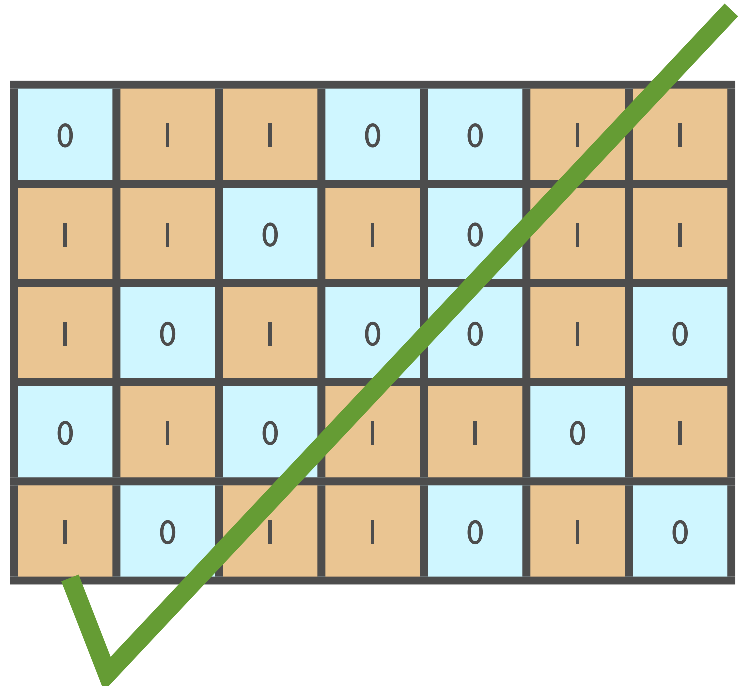

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Definition. *[**BBI**] For finite metric spaces (X, d_X) and (Y, d_Y), define the **Gromov-Hausdorff distance** between them by*

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2}\min_{C} \max_{(x,y),(x',y')\in C} |d_X(x,x') - d_Y(y,y')|$$

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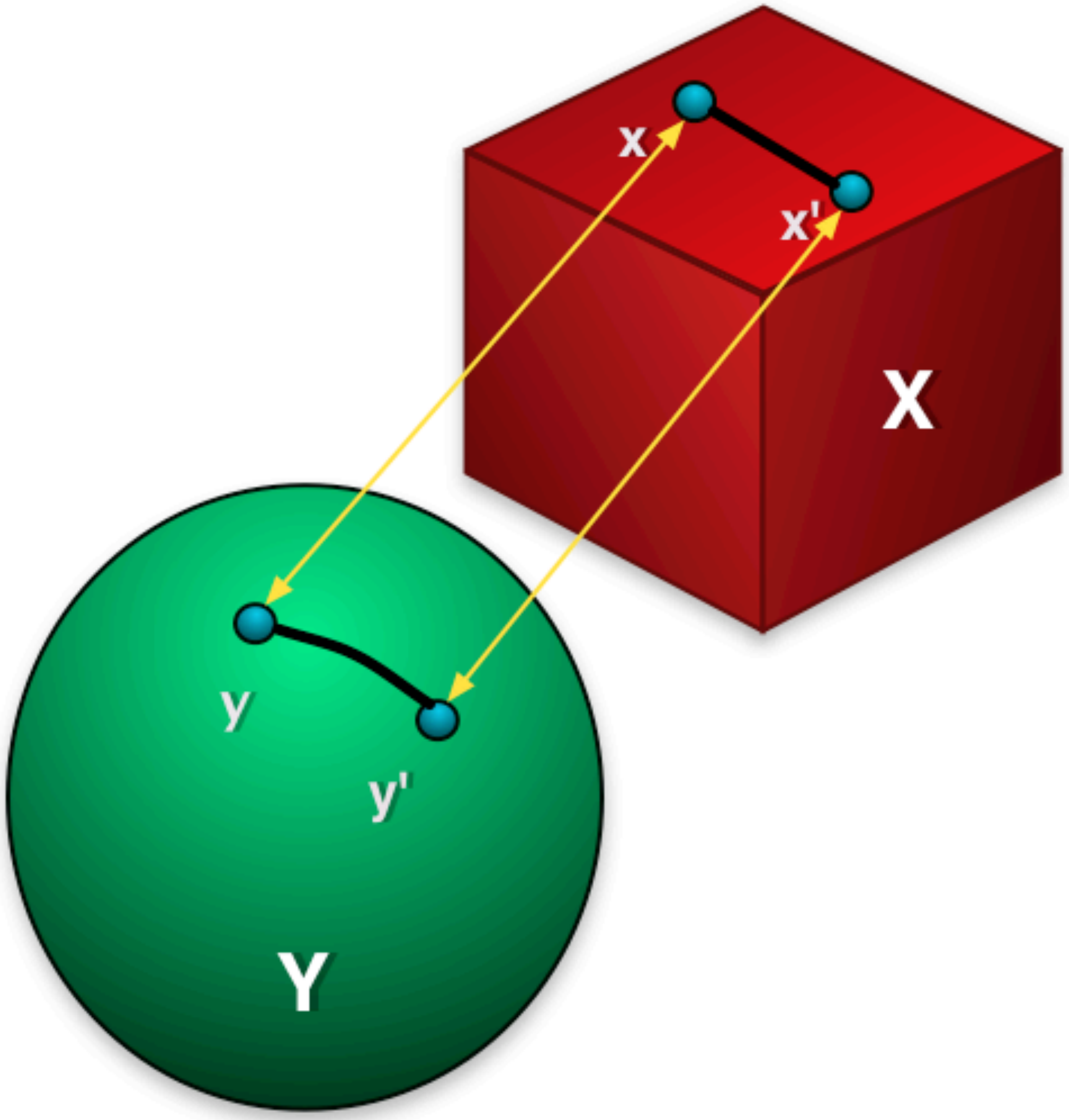
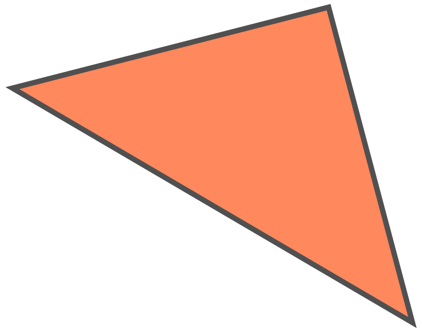


Construction of our signatures

- Our signatures take the form of **persistence diagrams**: we capture certain topological and metric information from the shape.

- First example: construction based on **Rips filtrations**: Let (X, d_X) be a shape.
  - Let $K_d(X)$ be the d-dimensional **full simplicial complex** on X.
  - To each $\sigma = [x_0, x_1, \ldots, x_k] \in K_d(X)$ assign its **filtration time**

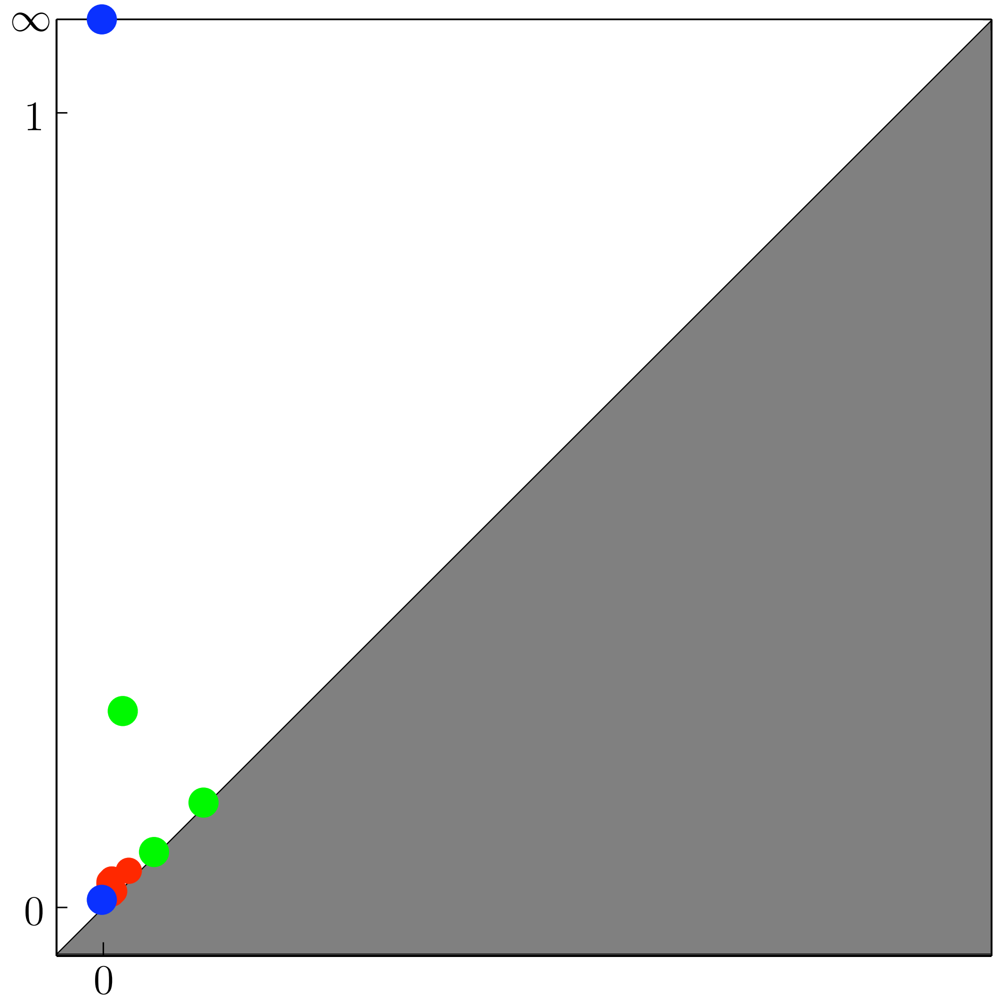
$$F(\sigma) := \frac{1}{2} \max_{i,j} d_X(x_i, x_j)$$

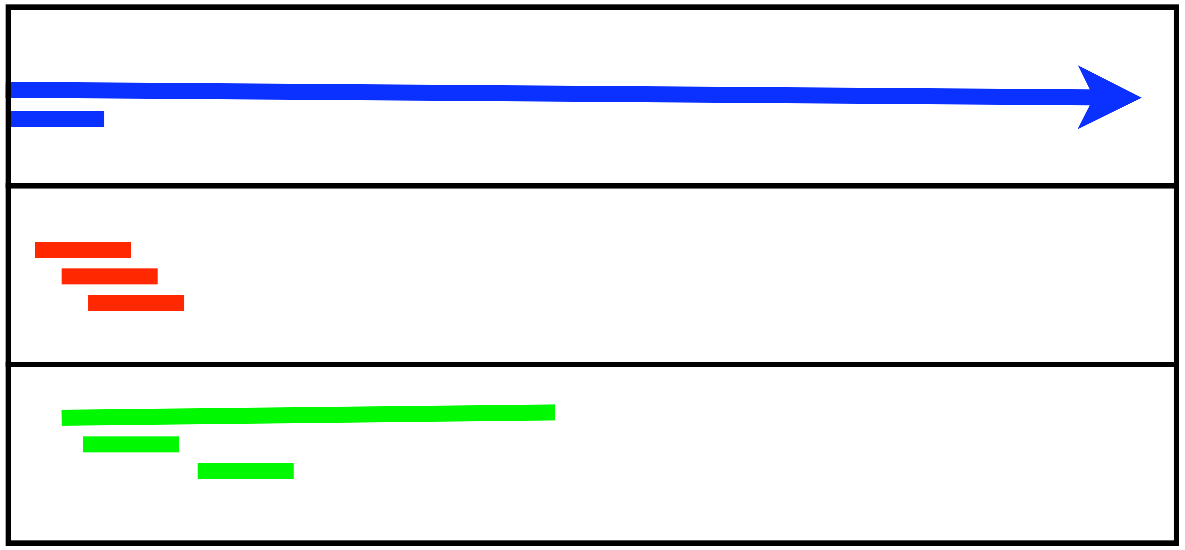


  - This gives rise to a **filtration** $\big(K_d(X), F\big)$.
  - Apply **persistence algorithm [ELZ00]** to summarize topological information in the filtration and obtain **persistence diagram**.

- Persistence diagrams are **colored multi-subsets** of the extended real plane.. can also be represented as **barcodes**.

- Let $\mathcal{D}$ denote the collection of all persistence diagrams. Compare two different persistence diagrams with **bottleneck distance** $\Longrightarrow$ view $\left(\mathcal{D}, d_B^\infty\right)$ as a metric space.





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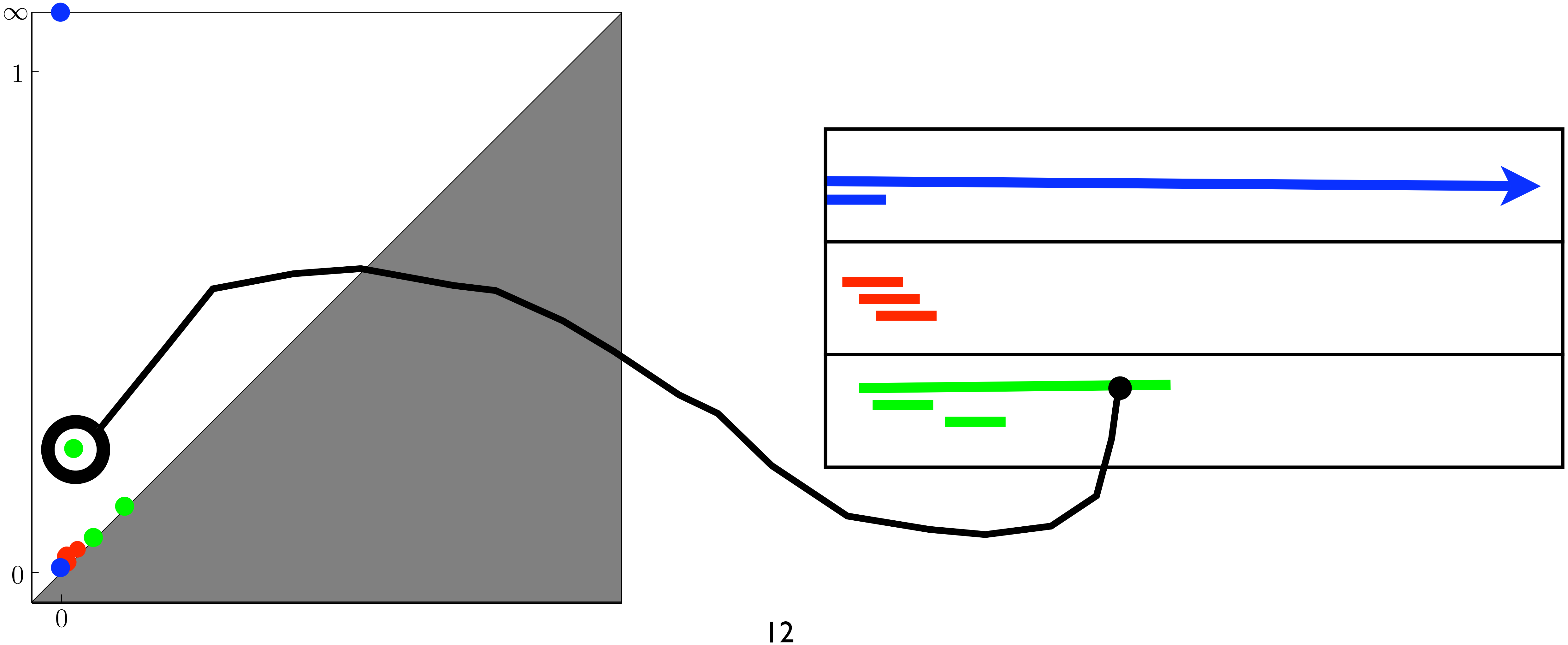

Example: Rips filtration on a torus

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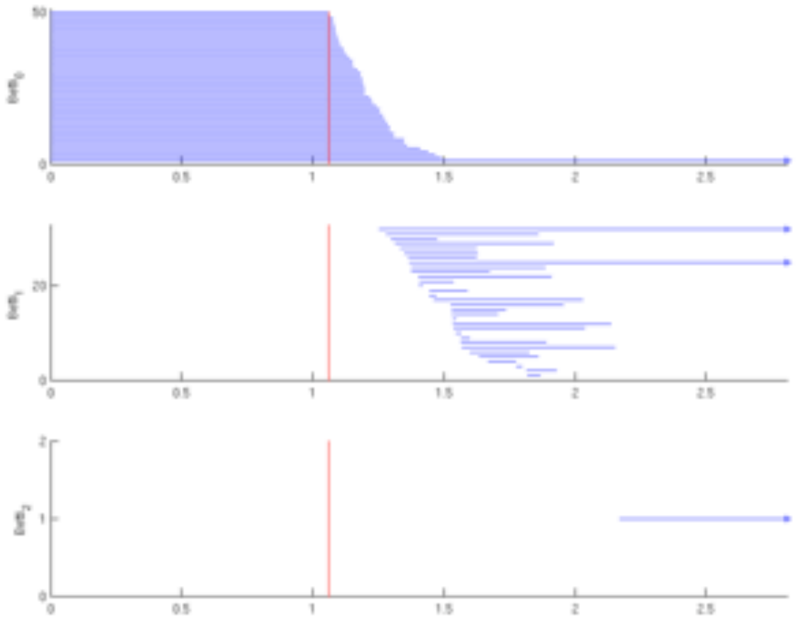
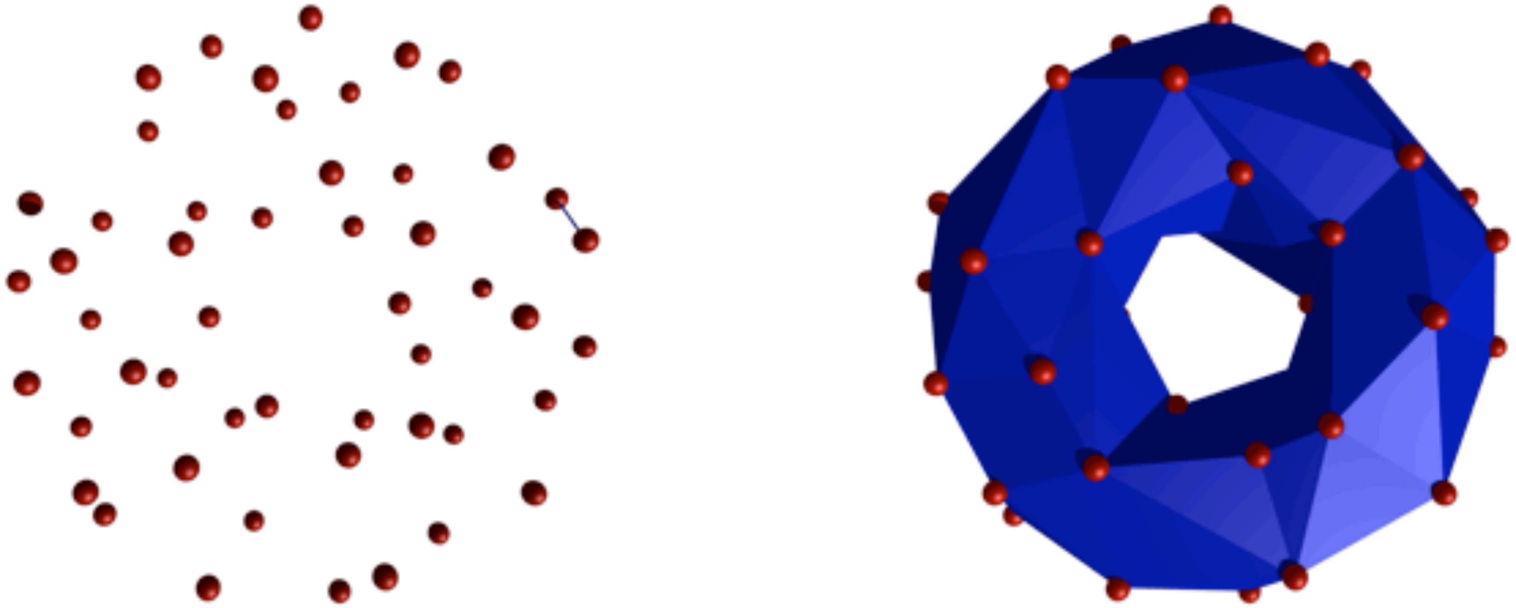




Our signatures: more richness

- Let's assume that in there is also a function defined on the shape: (X, d_X, f_X). Then, we redefine the filtration values of $\sigma = [x_0, x_1, \ldots, x_k]$

$$F(\sigma) = \max\left(\frac{1}{2}\max_{i,j} d_X(x_i, x_j), \max_i f_X(x_i)\right)$$

- Again, this gives rise to a **filtration**: $(K_d(X), F) \implies$ use **persistence algorithm** to obtain a persistence diagram.

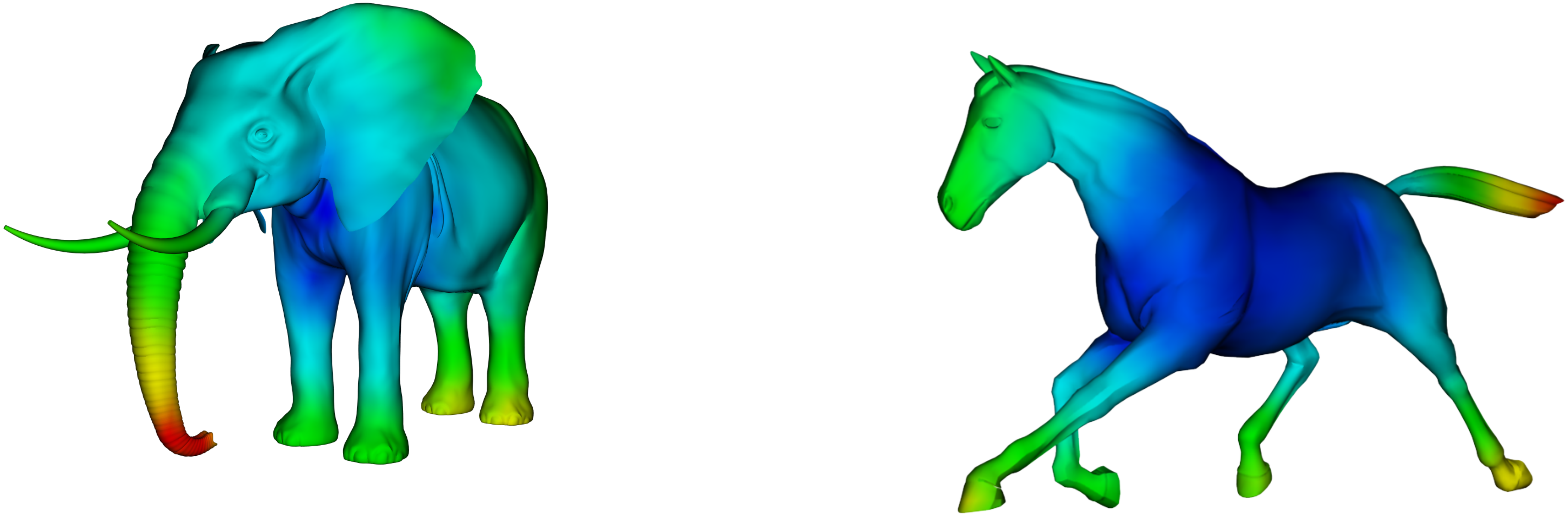
- This increases discrimination power!

- We denote by $\mathcal{H}$ a family of maps that attach a function to a given finite metric space.

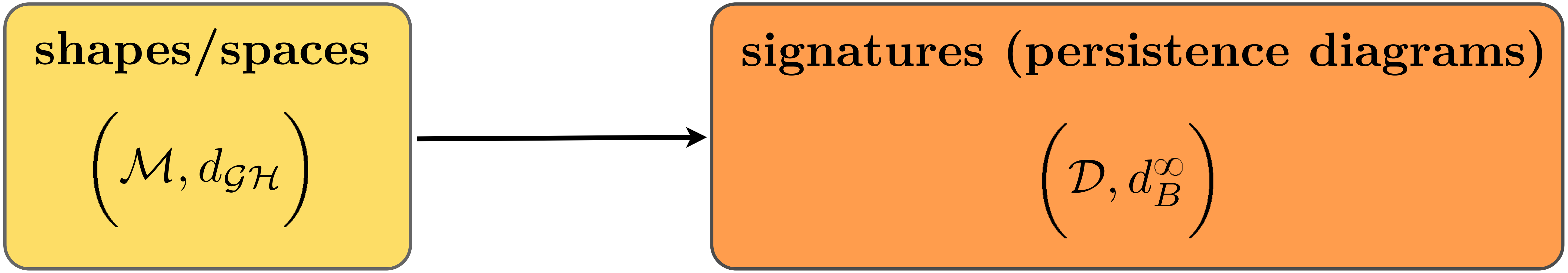
- Then, for each $h \in \mathcal{H}$, we denote by $D_h(X)$ the persistence diagram arising from the filtration above. This constitutes our family projection onto $\mathcal{D}$.

Example (Eccentricity)**.** *To each finite metric space* (X, d_X) *one can assign the* ***eccentricity*** *function:*

$$ecc_X(x) = \max_{x' \in X} d_X(x, x').$$



$h \in \mathcal{H}$



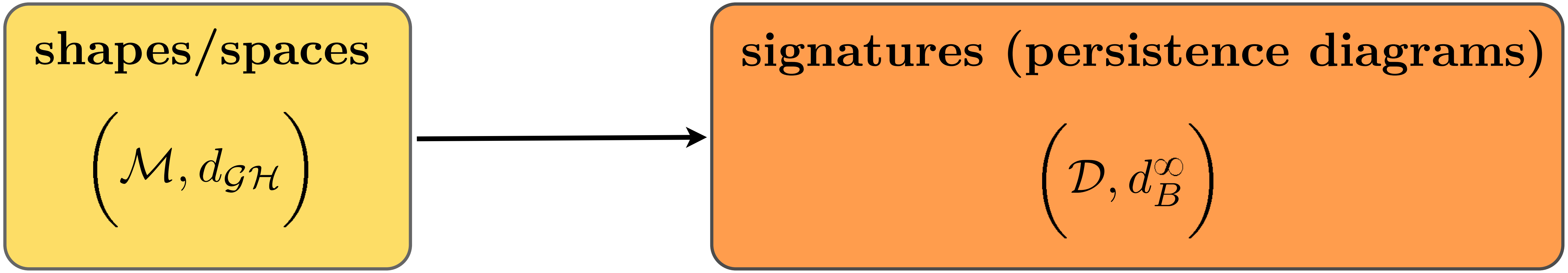
shapes/spaces

$$\Big(\mathcal{M}, d_{\mathcal{GH}}\Big)$$

signatures (persistence diagrams)

$$\Big(\mathcal{D}, d_B^\infty\Big)$$

$h \in \mathcal{H}$



shapes/spaces

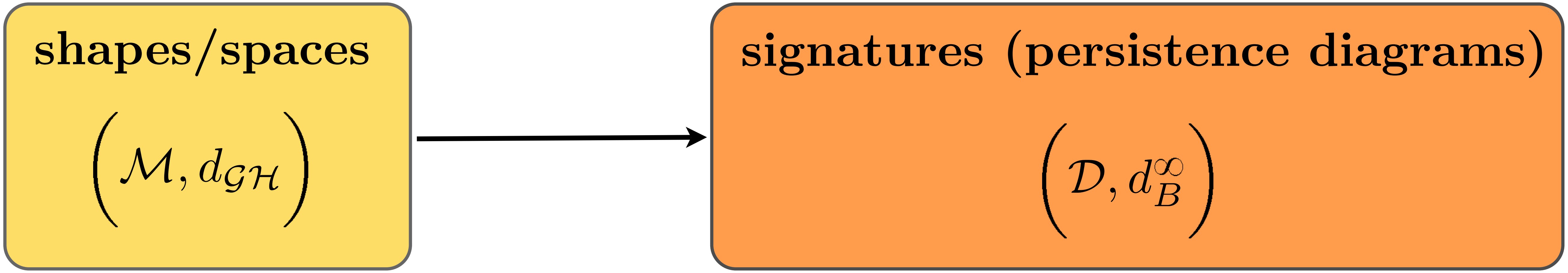
$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$

signatures (persistence diagrams)

$\left(\mathcal{D}, d_B^{\infty}\right)$

X, Y

$h \in \mathcal{H}$



shapes/spaces

$$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$$

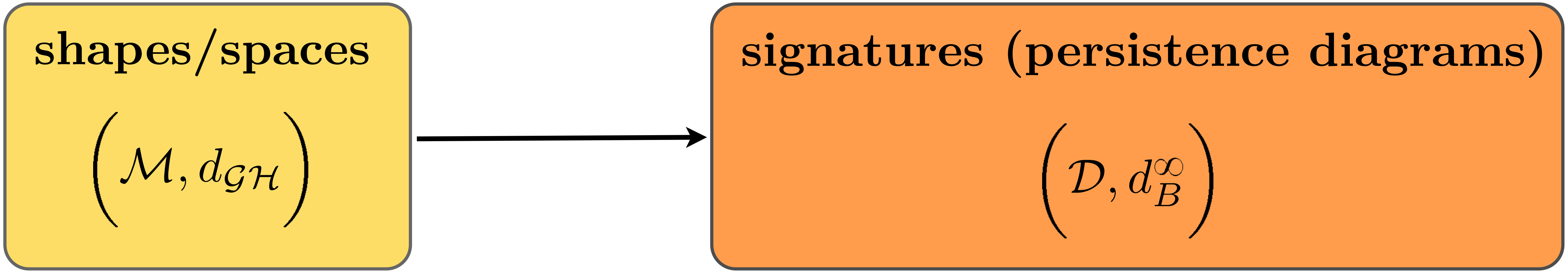
signatures (persistence diagrams)

$$\left(\mathcal{D}, d_B^{\infty}\right)$$

X, Y

$D_h(X),\ D_h(Y)$

$h \in \mathcal{H}$



shapes/spaces

$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$

signatures (persistence diagrams)

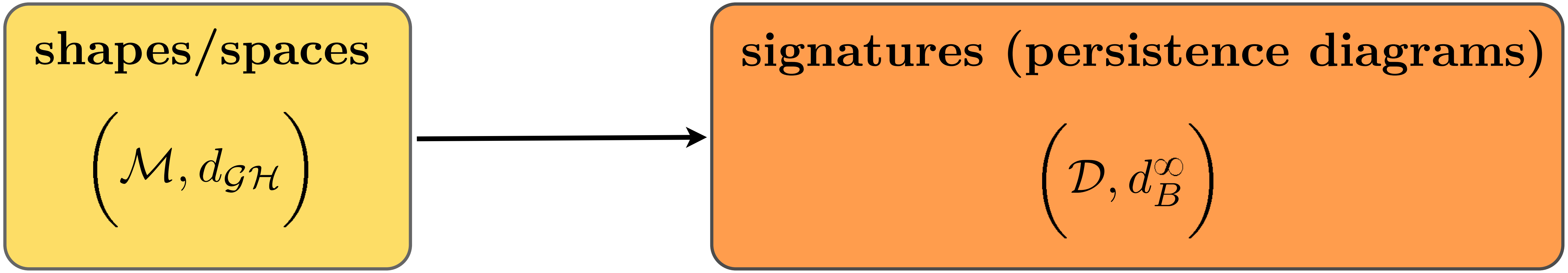
$\left(\mathcal{D}, d_B^\infty\right)$

X, Y

$D_h(X),\ D_h(Y)$

$$d_{\mathcal{GH}}(X, Y)$$

$h \in \mathcal{H}$



shapes/spaces

$$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$$

signatures (persistence diagrams)

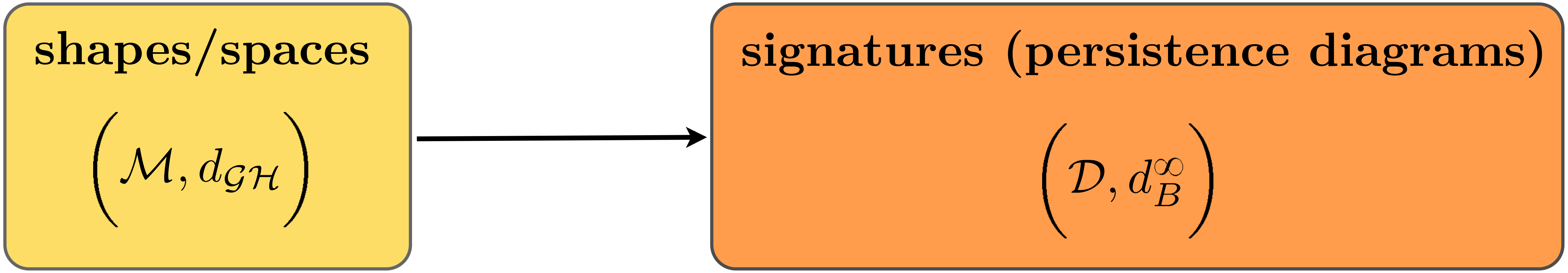
$$\left(\mathcal{D}, d_B^{\infty}\right)$$

X, Y

$D_h(X),\ D_h(Y)$

$$\xcancel{d_{\mathcal{GH}}(X, Y)}$$

$h \in \mathcal{H}$



shapes/spaces

$$\left(\mathcal{M}, d_{\mathcal{GH}}\right)$$

signatures (persistence diagrams)

$$\left(\mathcal{D}, d_B^\infty\right)$$

X, Y

$D_h(X),\ D_h(Y)$

$$\cancel{d_{\mathcal{GH}}(X, Y)}$$

$$\boxed{d_B^\infty(D_h(X),\ D_h(Y))}$$

Theorem (stability of our signatures). *For all* $X, Y \in \mathcal{M}$,

$$d_{\mathcal{GH}}\big(X, Y\big) \geqslant \qquad C(h) \cdot d_B^{\infty}\big(D_h(X), D_h(Y)\big).$$

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Theorem (stability of our signatures)**.** *For all* $X, Y \in \mathcal{M}$,

$$d_{\mathcal{GH}}\big(X, Y\big) \geqslant \sup_{h \in \mathcal{H}} C(h) \cdot d_B^{\infty}\big(D_h(X), D_h(Y)\big).$$

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Remark.

- *Proof relies on properties of the GH distance and new results on the stability of persistence diagrams [**CCGGO09**].*

- *For a given h, the computation leads to a **BAP** which can be solved in polynomial time.*

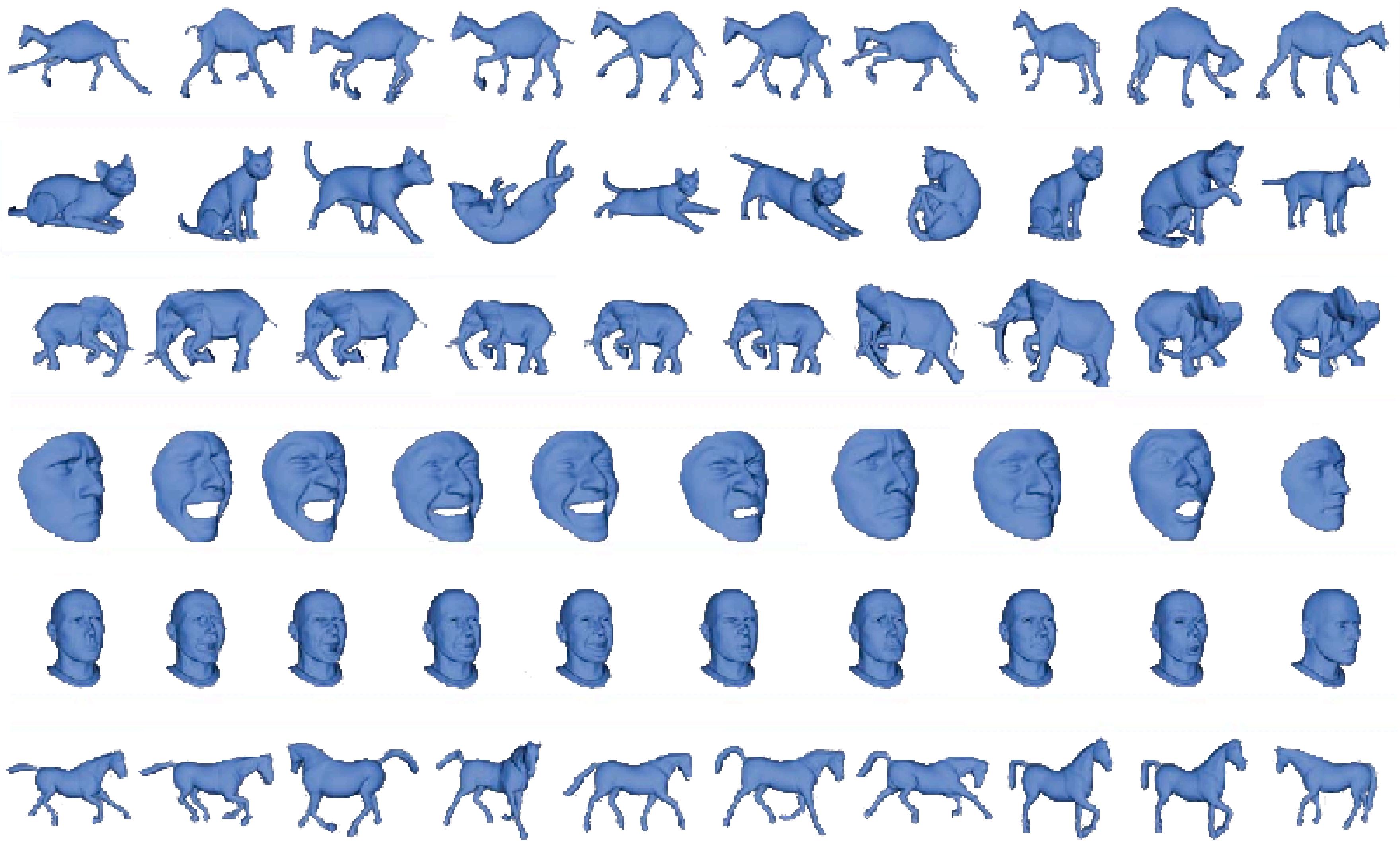
- *There are adaptations one can do in practice to speed up, see paper.*

- *One can obtain more generality and discrimination power by working in the class of **mm-spaces**: shapes are represented as triples* (X, d_X, μ_X) *where* μ_X *are **weights** assigned to each point see [**M07**] and paper.*

- *Our results include **stability of Rips persistence diagrams**.*

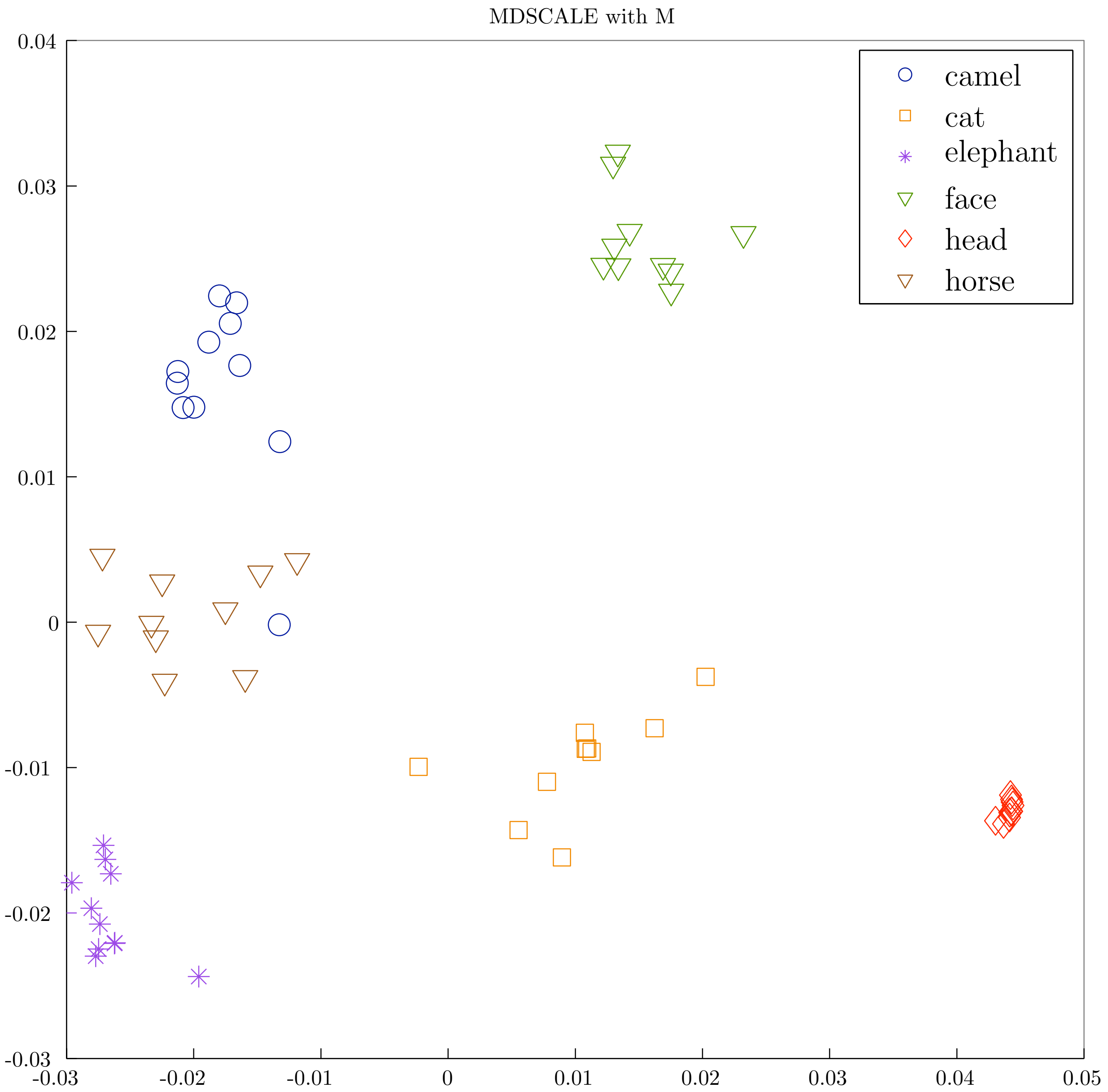
Some experiments

- Sumner database: 62 shapes total, 6 classes. Used graph estimate of the geodesic distance. Number of vertices ranged from 7K to 30K.



Some experiments

- Sumner database: 62 shapes total, 6 classes. Used graph estimate of the geodesic distance. Number of vertices ranged from 7K to 30K.
- Subsampled shapes and retained subsets of 300 points (farthest point sampling). Normalized distance matrices.
- Used the mm-space representation of shapes: weights were based on Voronoi regions.
- Used several functions $\pm\lambda \cdot h$ for λ in a finite subset of scales.
- Obtained 4% (or 2%) classification error in a 1-nn classification problem.


MDSCALE with M
camel
cat
elephant
face
head
horse
0.04
0.03
0.02
0.01
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-0.01
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0.03
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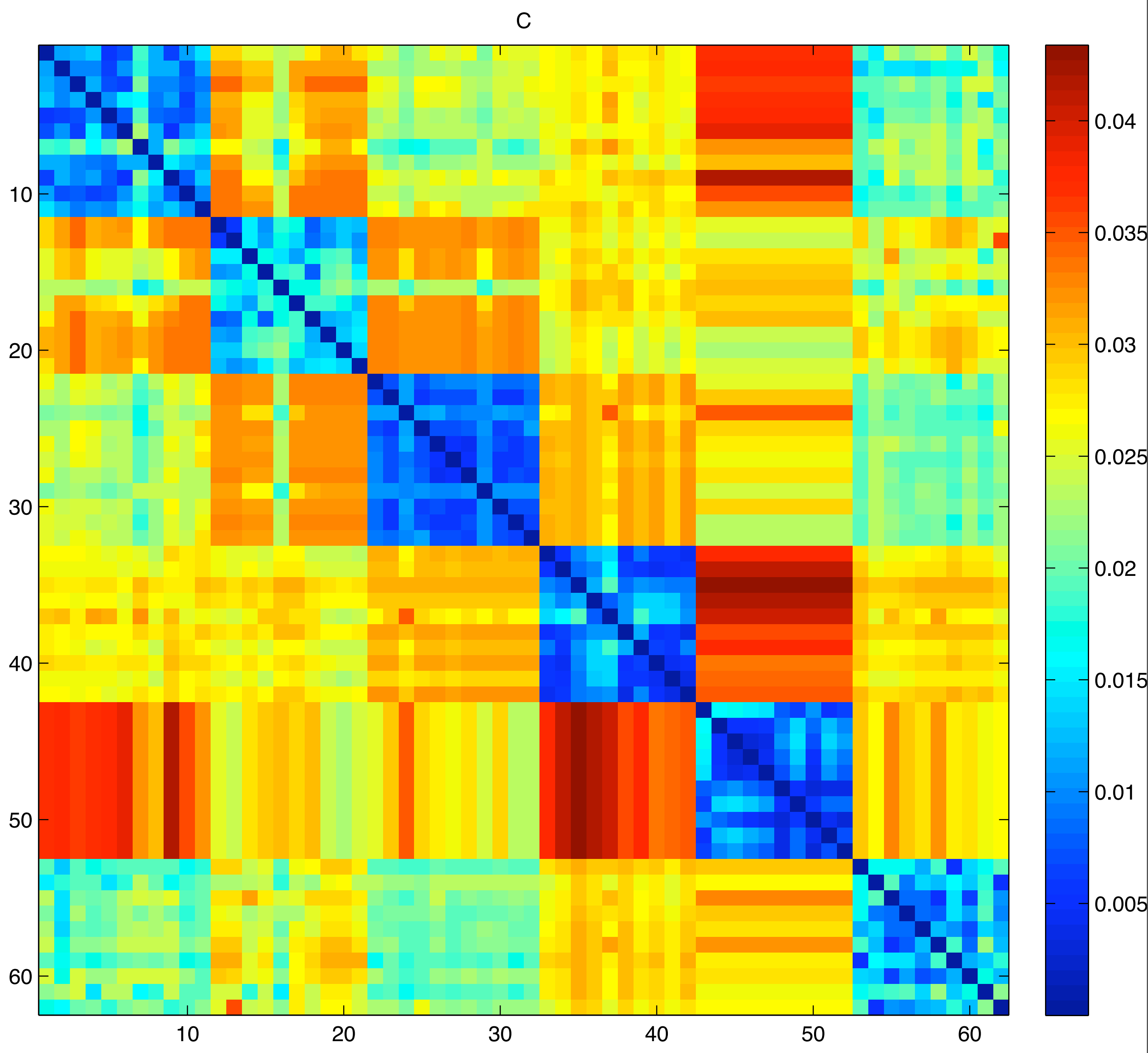
0.05


C
10
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0.03
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0.02
0.015
0.01
0.005

Discussion

- **Summary of our proposal**:
  - Use the metric (or mm-space) representation of shapes.
  - Formulate the shape matching problem using the Gromov-Hausdorff distance.
  - Compute our signatures for shapes.
  - Solve the BAP lower bounds: computationally easy! By our theorem, the computed quantities give lower bounds for the GH distance.
- **Implications and Future directions**:
  - We do not need a mesh– general: can be applied to any dataset.
  - We obtain stability of Rips persistence diagrams.
  - Richness of the family $\mathcal{H}$? how close can I get to the GH distance?
  - Local signatures: more discrimination.
  - Extension to partial shape matching: which (local) signatures are useful for this?

Acknowledgements

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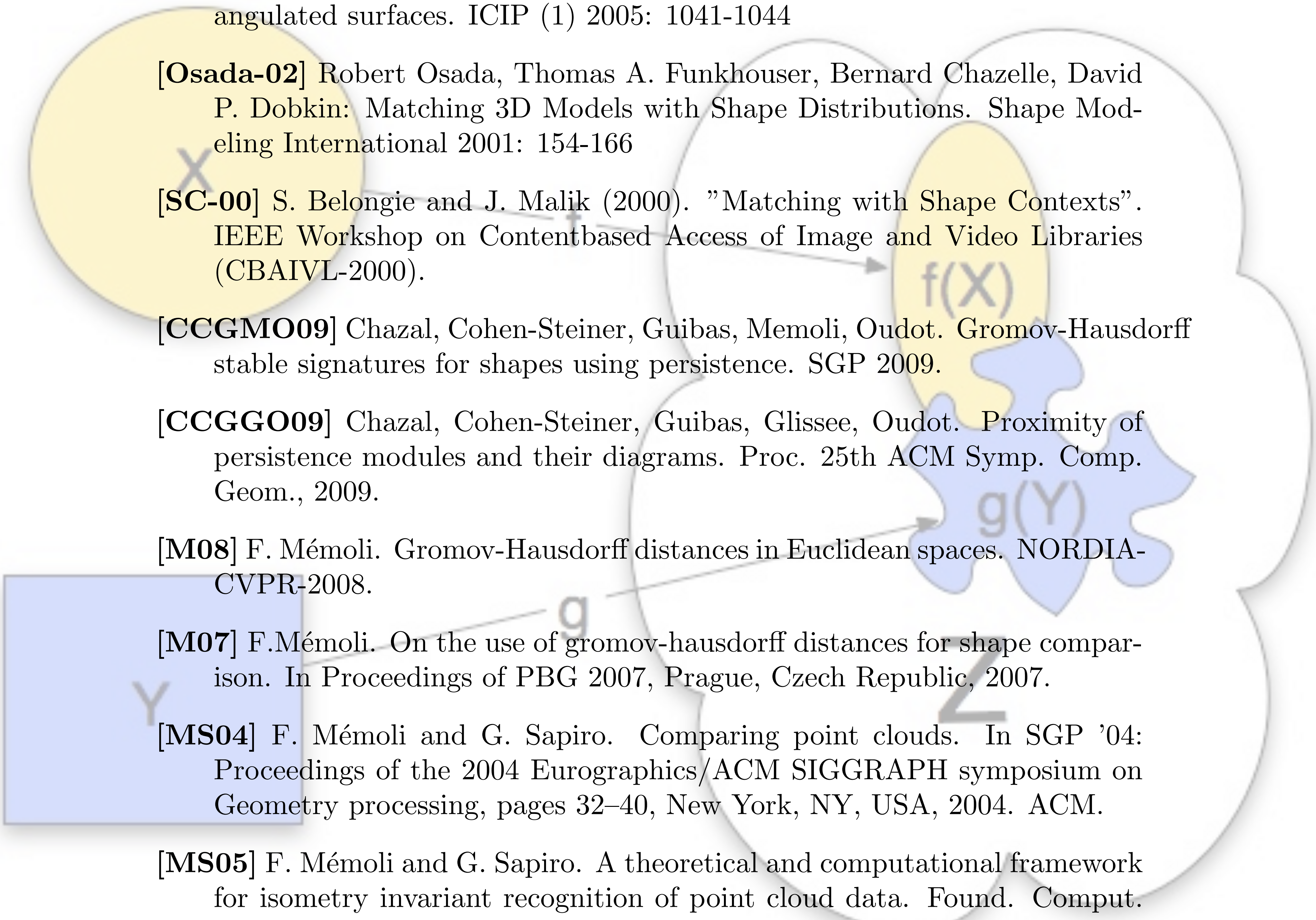
- INRIA-Stanford associated TGDA team.

Bibliography



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`http://math.stanford.edu/~memoli`

Let $X_1, X_2 \subset Z$ be two different samples of the same shape Z, and Y another shape then

$$\left| d_{\mathcal{GH}}(X_1, Y) - d_{\mathcal{GH}}(X_2, Y) \right| \leqslant d_{\mathcal{GH}}(X_1, X_2) \leqslant r_1 + r_2$$