Coupling On-line and Off-line Random Graphs

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March 1st

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- Several **random graph models**
- The method to analyze them
  (Especially, by relating one random graph to another random graph)

Reminder

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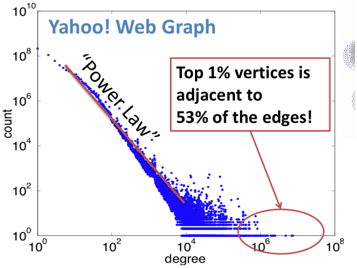
  - Natural Graphs:



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e.g. $G(n, n^{-0.9})$ almost surely contains triangle.

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The probability distribution of the random graph depends upon the choice of the model.

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The on-line model can be viewed as an infinite sequence of off-line models where the random graph model at time t may depend on all the earlier decisions.

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The on-line models are **harder to analyze** than the off-line models, but **closer** to the way that realistic networks are generated.

We analyze the on-line models using the knowledge that we have about the off-line models.

Our goal is **to couple the on-line model with the off-line model of random graphs with a similar power law degree distribution** so that we can apply the techniques from the off-line model to the on-line model.

Graph property

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Examples

- The property of containing the complete graph K_3

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- The property of containing the complete graph K_3
- The property of being connected (Non-example)

Dominance

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In this case, we write

$$G_1 \geq G_2.$$

Dominance

e.g. For any $p_1 \leq p_2$, $G(n, p_1) \leq G(n, p_2)$

Dominance

Definition

For any $\epsilon > 0$, we say G_1 dominates G_2 with an error estimate ϵ if

$$Pr(G_1 \text{ satisfies } A) + \epsilon \geq Pr(G_2 \text{ satisfies } A)$$

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e.g. For any $\delta > 0$, we have almost surely

$$G(n, (1-\delta)\frac{m}{\binom{n}{2}}) \preceq F(n, m) \preceq G(n, (1+\delta)\frac{m}{\binom{n}{2}})$$

Edge-independent

Definition

A random graphs G is called **edge-independent** if there is an edge-weighted function $p : E(K_n) \to [0, 1]$ satisfying

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For any given random graph model, **it would be advantageous if we can establish some comparisons with edge-independent random graph**

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Vertex-growth step: Add a new vertex v and form a new edge from v to an existing vertex u chosen with probability proportional to an existing vertex u chosen with probability proportional to d_u

Edge-growth step: Add a new edge with endpoints to be chosen among existing vertices with probability proportional to the degrees. If it already exists in the current graph, the generated edge is discarded. The edge-growth step is repeated until a new edge is successfully added.

A Growth-Deletion Model for Random Power Law Graphs

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Edge-deletion step: Delete an edge randomly.

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For non-negative values p_1, p_2, p_3, p_4 summing to 1, we consider the following growth-deletion model $G(p_1, p_2, p_3, p_4)$:

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Otherwise, with probability $p_4 = 1 - p_1 - p_2 - p_3$, take an edge-deletion step.

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- We assume $p_3 < p_1$ and $p_4 < p_2$ so that the number of vertices and edge grows as t goes to infinity.

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- We assume $p_3 < p_1$ and $p_4 < p_2$ so that the number of vertices and edge grows as t goes to infinity.
- If $p_3 = p_4 = 0$, the model is the usual preferential attachment model which generates power law graphs with exponent $\beta = 2 + \frac{p_1}{p_1 + 2p_2}$.

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Suppose $p_3 < p_1$ and $p_4 < p_2$. Then almost surely the degree sequence of the growth-deletion model $G(p_1, p_2, p_3, p_4)$ follows the power law distribution with the exponent

$$\beta = 2 + \frac{p_1 + p_3}{p_1 + 2p_2 - p_3 - 2p_4}$$

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- For p_i's in certain ranges, this value can be below 1 and the random graph is not connected.

$\Longrightarrow$ We consider the modified model $G(p_1, p_2, p_3, p_4, m)$ for some integer m which will generate random graphs which have expected degree $m\frac{(p_1 + p_2 - p_4)}{(p_1 + p_3)}$.

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- Then almost surely the degree sequence of the growth-deletion model $G(p_1, p_2, p_3, p_4, m)$ follows the power law distribution with the exponent β being the same as the exponent for the model $G(p_1, p_2, p_3, p_4)$.
- Many results for $G(p_1, p_2, p_3, p_4, m)$ can be derived in the same fashion as for $G(p_1, p_2, p_3, p_4)$

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For any two vertices u and v, let $p_{uv}^{(i)}$ be the probability of edge uv in G_i for $i = 1, 2$. We have

$$p_{uv}^{(1)} = (1 - o(1))p_{uv}^{(2)}$$

The main theorem 1: Fan Chung and Linyuan Lu, 2004

Suppose $p_3 < p_1, \quad p_4 < p_2 \quad and \quad \log n \ll m < t^{\frac{p_1}{2(p_1+p_2)}}$. Then,

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Suppose $p_3 < p_1, \quad p_4 < p_2 \quad and \quad \log n \ll m < t^{\frac{p_1}{2(p_1+p_2)}}$. Then,

(1) Almost surely the degree sequence of the growth-deletion model $G(p_1, p_2, p_3, p_4, m)$ follows the power law distribution with the exponent

$$\beta = 2 + \frac{p_1 + p_3}{p_1 + 2p_2 - p_3 - 2p_4}$$

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Suppose $p_3 < p_1, \quad p_4 < p_2 \quad and \quad \log n \ll m < t^{\frac{p_1}{2(p_1+p_2)}}$. Then,

(2) $G(p_1, p_2, p_3, p_4, m)$ is almost surely edge-independent. It dominates and is dominated by an edge-independent graph with probability $p_{ij}^{(t)}$ of having an edge between vertices i and j, $i < j$, at time t, satisfying:

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$$p_{ij}^{(t)} \approx \begin{cases} \frac{p_2 m}{2p_4\tau(2p_2-p_4)} \frac{t^{2\alpha-1}}{i^\alpha j^\alpha}\left(1+\left(1-\frac{p_4}{p_2}\right)\left(\frac{j}{t}\right)^{\frac{1}{2r}+2\alpha-1}\right), & \text{if } \; i^\alpha j^\alpha \gg \frac{p_2 m t^{2\alpha-1}}{4\tau^2 p_4} \\ 1-(1+o(1))\frac{2p_4\tau}{p_2 m} i^\alpha j^\alpha t^{1-2\alpha}, & \text{if } \; i^\alpha j^\alpha \ll \frac{p_2 m t^{2\alpha-1}}{4\tau^2 p_4} \end{cases}$$

where $\alpha = \frac{p_1(p_1+2p_2-p_3-2p_4)}{2(p_1+p_2-p_4)(p_1-p_3)}$ and $\tau = \frac{(p_1+p_2-p_4)(p_1-p_3)}{p_1+p_3}$

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Without the assumption on m, we have the following general but weaker result. We say the index of a vertex u is i if u is generated at time i.

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In $G(p_1, p_2, p_3, p_4, m)$ with $p_3 < p_1, \quad p_4 < p_2$, let S be the set of vertices with index i satisfying

$$i \gg m^{\frac{1}{\alpha}} t^{1-\frac{1}{2\alpha}}.$$

Let G_S be the induced subgraph of $G(p_1, p_2, p_3, p_4, m)$ on S. Then we have

(1) G_S dominates a random power law graph G_1, in which the expected degrees are given by

$$w_i \approx \frac{p_2 m}{2p_4\tau(2p_2 - p_4)(\frac{p_1}{p_1-p_3} - \alpha)} \frac{t^\alpha}{i^\alpha}$$

The Main theorem 2: Fan Chung and Linyuan Lu, 2004

(2) G_S is dominated by a random power law graph G_2, in which the expected degrees are given by

$$w_i \approx \frac{m}{2p_4\tau(\frac{p_1}{p_1-p_3}-\alpha)}\frac{t^\alpha}{i^\alpha}.$$

The Main theorem 3: Fan Chung and Linyuan Lu, 2004

In $G(p_1, p_2, p_3, p_4, m)$ with $p_3 < p_1$, $p_4 < p_1$, let T be the set of vertices with index i satisfying

$$i \ll m^{\frac{1}{\alpha}} t^{1-\frac{1}{2\alpha}}.$$

Then the induced subgraph G_T of $G(p_1, p_2, p_3, p_4, m)$ is almost a complete graph. Namely, G_T dominates an edge-independent an edge-independent graph with $p_{ij} = 1 - o(1)$

Ingredient of Proof for the Main Theorems

- The basic idea : the martingale method
- But with substantial difference

- A martingale involves a sequence of functions with consecutive functions having small bounded differences, each function is defined on a fixed probability space Ω.

- For the on-line model, the probability space for the random graph generated at each time instance is different in general. (We have a sequence of probability spaces where two consecutive ones have "small" difference.)

Bibliography

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