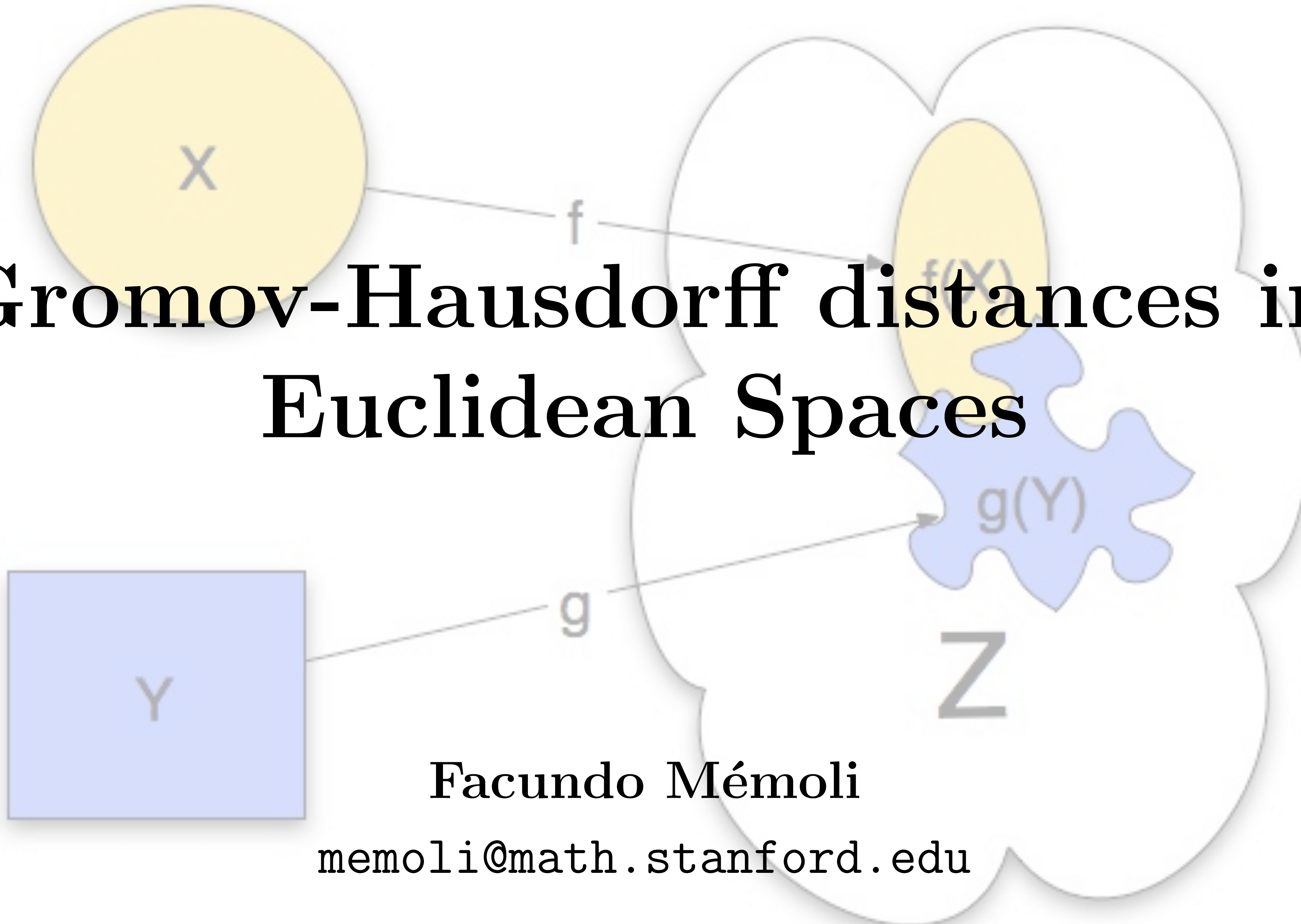
Gromov-Hausdorff distances in Euclidean Spaces



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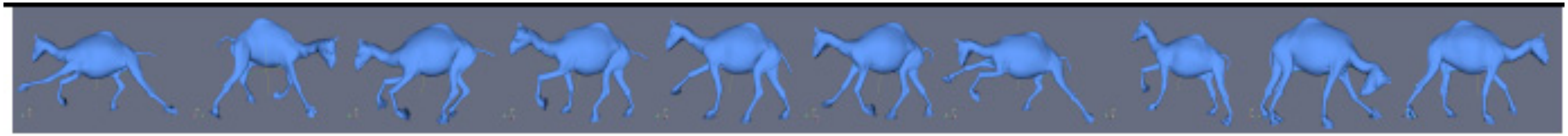

The GH distance for Shape Comparison

- Regard shapes as (compact) metric spaces. Let $\mathcal{X}$ denote set of all compact metric spaces. Define metric on $\mathcal{X}$, then $(\mathcal{X}, d_{\mathcal{GH}})$ is itself a metric space.

- The metric with which one endows the shapes depends on the desired invariance. For example, if invariance to
  - *rigid isometries* is desired, use Euclidean distance.
  - *bends* is desired, use "intrinsic" distance.

- GH distance provides reasonable framework for Shape Comparison: good theoretical properties.

Properties of GH distance:

1. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces then

$$d_{\mathcal{GH}}(X, Y) \leq d_{\mathcal{GH}}(X, Z) + d_{\mathcal{GH}}(Y, Z).$$

2. If $d_{\mathcal{GH}}(X, Y) = 0$ and (X, d_X), (Y, d_Y) are compact metric spaces, then (X, d_X) and (Y, d_Y) are isometric.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a finite subset of the compact metric space (X, d_X). Then,

$$d_{\mathcal{GH}}(X, \mathbb{X}_n) \leq d_{\mathcal{H}}(X, \mathbb{X}_n).$$

4. For compact metric spaces (X, d_X) and (Y, d_Y):

$$\begin{aligned} \frac{1}{2} |\mathrm{diam}(X) - \mathrm{diam}(Y)| &\leq d_{\mathcal{GH}}(X, Y) \\ &\leq \frac{1}{2} \max\left(\mathrm{diam}(X), \mathrm{diam}(Y)\right) \end{aligned}$$

In Euclidean spaces...

For $X, Y \subset \mathbb{R}^n$, we endow them with the Euclidean metric to form metric spaces $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$. Then, we have two possibilities:

$$d^{\mathbb{R}^n}_{\mathcal{H},iso}(X,Y) \quad \text{vs.} \quad d_{\mathcal{GH}}(X,Y)$$

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For $X, Y \subset \mathbb{R}^n$, we endow them with the Euclidean metric to form metric spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$. Then, we have two possibilities:

$$\mathbf{EH} = d_{\mathcal{H},iso}^{\mathbb{R}^n}(X,Y) \quad \text{vs.} \quad d_{\mathcal{GH}}(X,Y) = \mathbf{GH}$$

- **EH** is the usual choice.. [GMO99]
- The works of [MS04,MS05] and [BBK06] raise the question of whether one could use the **GH** distance for matching sets in $\mathbb{R}^n$ under Euclidean isometries
- Note that as n increases there may be some gain in using **GH** instead of **EH** (complexity of computing **GH** doesn't depend on n).
- Very important from Theoretical point of view: helps understanding more about the landscape of different metrics for shapes and their different properties and inter-relationships.

What we are going to prove: spoiler

1. For all (compact) Euclidean metric spaces:

$$\mathbf{GH} \leq \mathbf{EH}.$$

2. Equality above doesn't hold in general: there exist sets in $\mathbb{R}^n$ for which

$$\mathbf{GH} < \mathbf{EH}.$$

3. What about bounding $\mathbf{EH} \leq C \cdot \mathbf{GH}^t$ for some constant $C = C(n)$ and some $t > 0$? In this respect, for any $\varepsilon > 0$, we find subsets of $\mathbb{R}^2$ for which

$$\mathbf{EH} \geq \sqrt{\varepsilon/2} \quad \text{and} \quad \mathbf{GH} \leq \varepsilon$$

so $t = 1$ is not achievable in general!

4. In general, for all (compact) Euclidean metric spaces:

$$\mathbf{GH} \leq \mathbf{EH} \leq C(n) \cdot \mathbf{GH}^{\frac{1}{2}}$$

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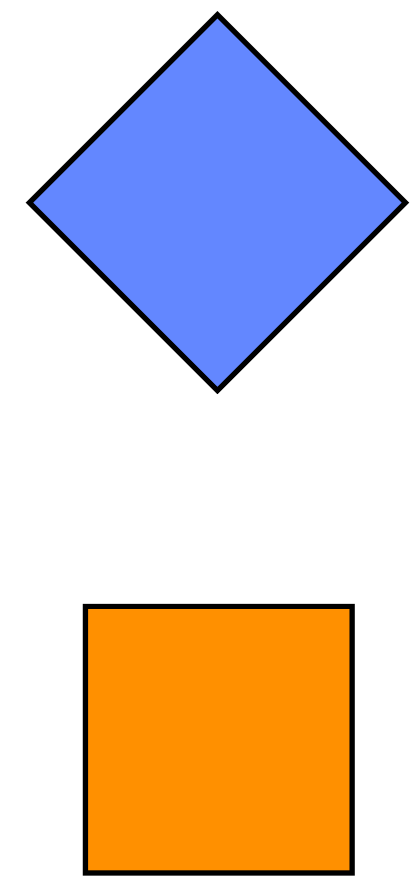
Background concepts

- Let $E(n)$ denote the group of Euclidean isometries in $\mathbb{R}^n$.
- **EH** $= d^{\mathbb{R}^n}_{\mathcal{H},iso}(X,Y)$

$$:= \inf_{T \in E(n)} d^{\mathbb{R}^n}_{\mathcal{H}}(X, T(Y)).$$

- **GH** admits several equivalent expressions:



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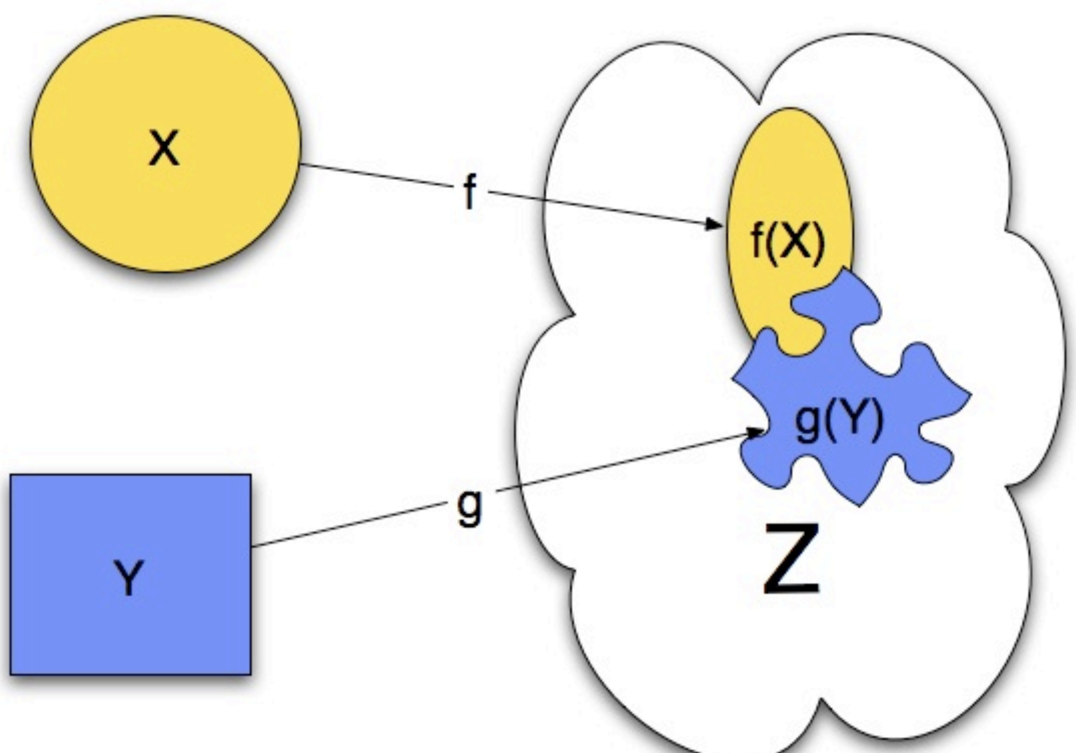
GH: original definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



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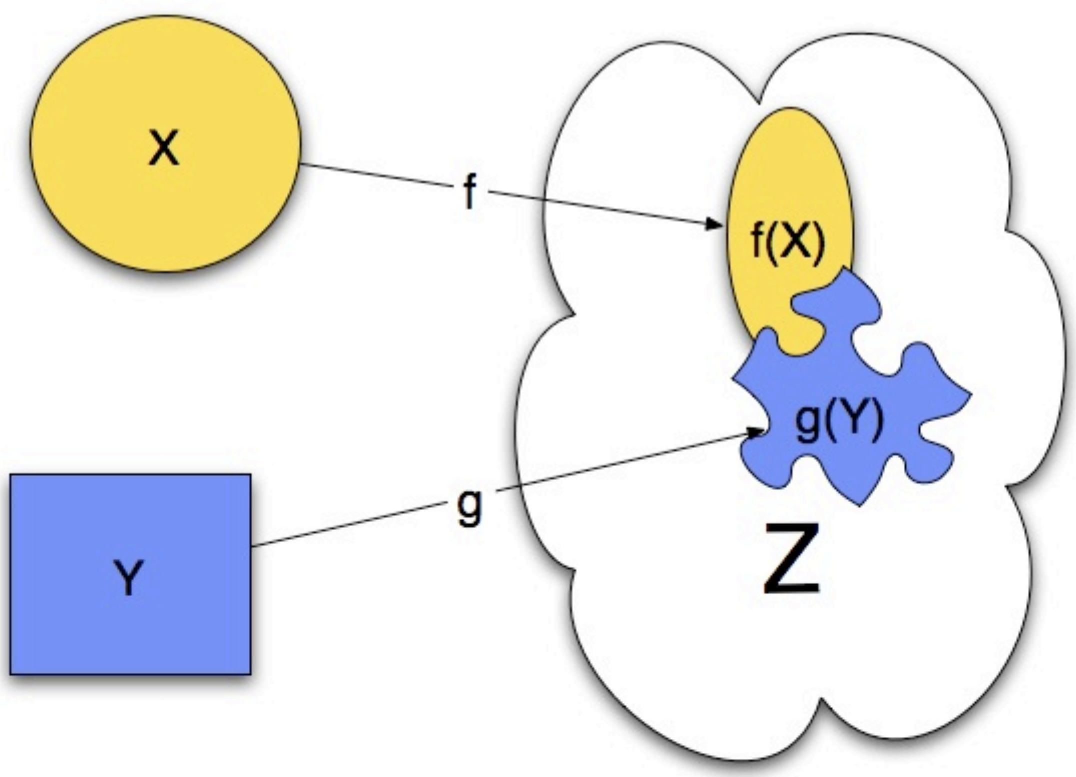

Notice that when X, Y are Euclidean, one can take $Z = \mathbb{R}^n$ and hence

$$\mathbf{GH}(X,Y) \leq \mathbf{EH}(X,Y).$$

GH: alternative expression

It is enough to consider $Z = X \sqcup Y$ and then we obtain

$$d_{\mathcal{GH}}(X,Y) = \inf_{d} d_{\mathcal{H}}^{(Z,d)}(X,Y)$$

where d is a metric on $X \sqcup Y$ that reduces to d_X and d_Y on $X \times X$ and $Y \times Y$, respectively. Denote by $\mathcal{D}(d_X, d_Y)$ the set of all such metrics.

$$\begin{array}{cc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} d_X & \mathbf{D} \\ \mathbf{D}^T & d_Y \end{pmatrix} = d \end{array}$$

In other words: you need to **glue** X and Y in an optimal way: you need to minimize

$$J(\mathbf{D}) := \max(\max_x \min_y \mathbf{D}(x,y), \max_y \min_x \mathbf{D}(x,y)).$$

Note that $\mathbf{D}$ consists of $n_X \times n_Y$ positive reals that must satisfy $\sim n_X \cdot C_2^{n_Y} + n_Y \cdot C_2^{n_X}$ linear constraints.

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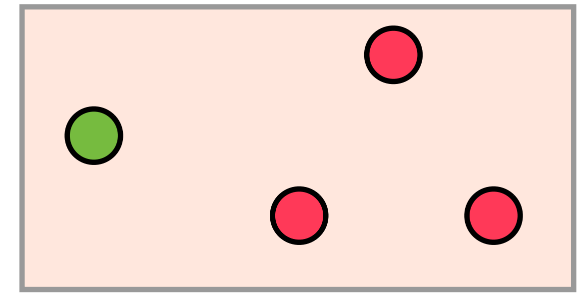


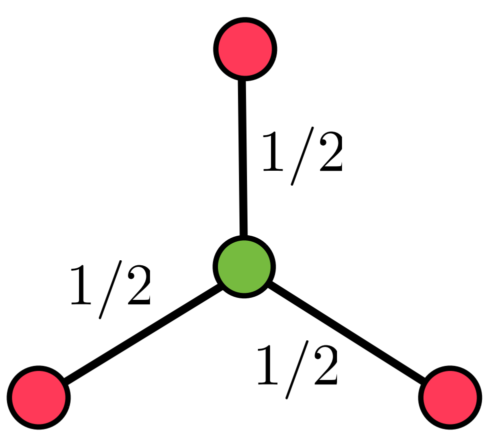
Question 1: Is EH = GH in general?

Answer is **no**.

Consider X and Y in $\mathbb{R}^2$ given by $X = \{p\}$ and $Y = \{y_1, y_2, y_3\}$, where y_i $i = 1, 2, 3$ are vertices of an equilateral triangle with side length 1.

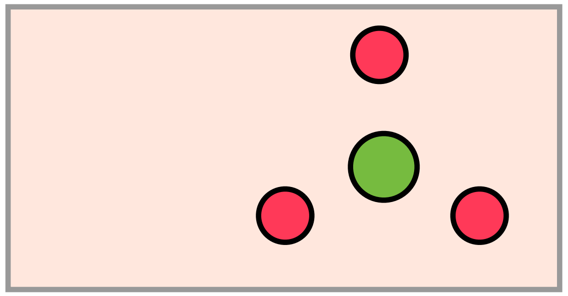
In this case

- **EH** $= \frac{1}{\sqrt{3}}$. Indeed, the optimal Euclidean isometry takes p into the center of the triangle.
- **GH** $= \frac{1}{2}$. Indeed, the optimal space Z is a tree-like metric space. Alternatively the optimal metric on $X \sqcup Y$ is



Z

$$\begin{array}{c|cccc} & p & y_1 & y_2 & y_3 \\ \hline p & 0 & 1/2 & 1/2 & 1/2 \\ y_1 & 1/2 & 0 & 1 & 1 \\ y_2 & 1/2 & 1 & 0 & 1 \\ y_3 & 1/2 & 1 & 1 & 0 \end{array} = d$$



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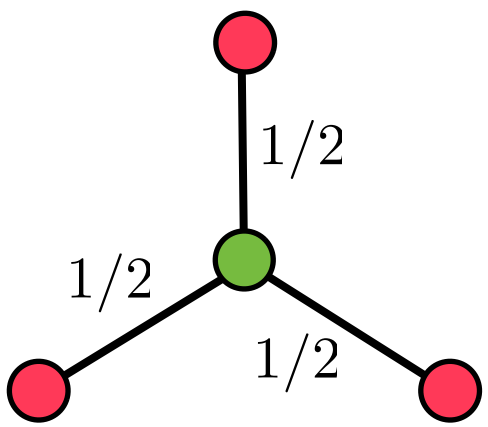

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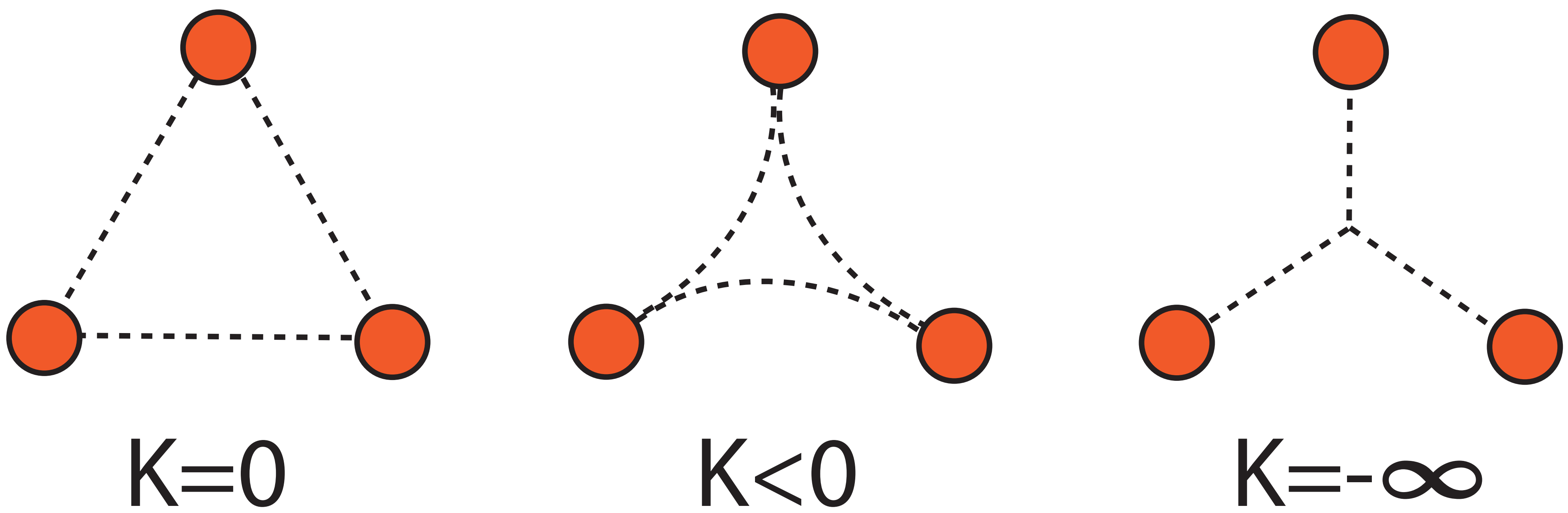
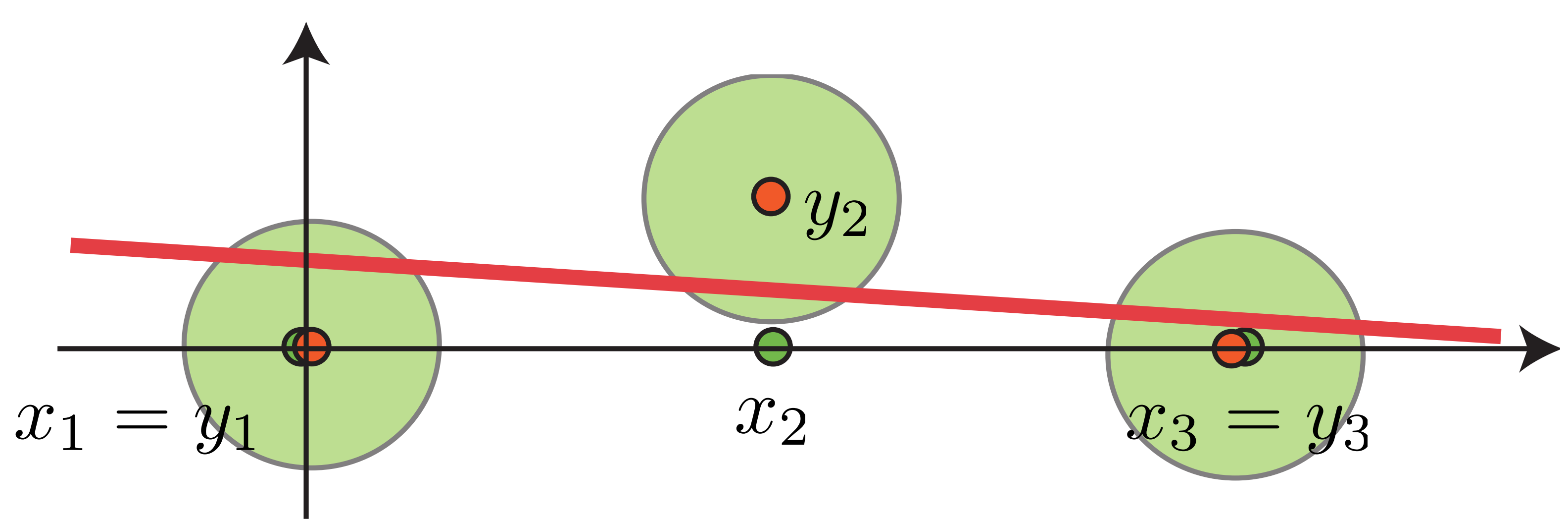

K=0
K<0
K=-∞

Question 2: What is the maximal t s.t. $\mathbf{EH} \leq C \cdot \mathbf{GH}^t$?

Answer is $t \leq 1/2$.

Pick $\varepsilon > 0$. Consider X and Y as in the figure (X is in green, Y in red). The y-coordinate of y_2 equals $h := \sqrt{2\varepsilon}$.

- It is easy to check that $\mathbf{GH} \leq \varepsilon$.
- Let $\mathbf{EH} = \alpha$. Consider the light green balls of radius α around each y_i.
- Let T be the Euclidean isometry s.t. $d_{\mathcal{H}}^{\mathbb{R}^2}(T(X), Y) = \alpha$.
- T must map the x-axis into a line (the red line in the figure) intersecting the three balls (otherwise, one of the y_i wouldn't have a point in x within distance α). This forces $2\alpha \geq h$. This means, $\alpha \geq \sqrt{\varepsilon/2}$.
- We've found $\mathbf{GH} \leq \varepsilon$ and $\mathbf{EH} \geq \sqrt{\varepsilon/2}$.



Question 3: does $t = 1/2$ work in general?

Answer is **yes**!

Theorem 1. *Let $X, Y \subset \mathbb{R}^n$ be compact. Then,*

$$d_{\mathcal{GH}}(X,Y) \leq d^{\mathbb{R}^n}_{\mathcal{H},iso}(X,Y) \leq c_n \cdot M^{\frac{1}{2}} \cdot (d_{\mathcal{GH}}(X,Y))^{\frac{1}{2}}$$

where $M = \max(diam(X), diam(Y))$ and c_n is a constant that depends only on n.

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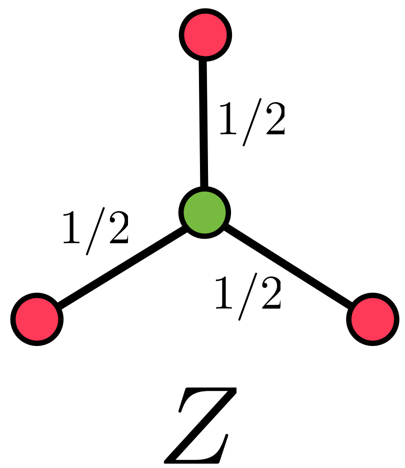
What is the source of the gap?

The problem is that we are allowing all the gluing metrics to be 'outside' of the set of metrics that can be *realized* in Euclidean spaces. In our first counterexample, d cannot be realized in any Euclidean space!



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Closing the gap

Remember that

$$d_{\mathcal{GH}}(X,Y) = \inf_d d_{\mathcal{H}}^{(X \sqcup Y, d)}(X,Y)$$

where d is a metric on $X \sqcup Y$ that reduces to d_X and d_Y on $X \times X$ and $Y \times Y$, respectively. Denote by $\mathcal{D}(d_X, d_Y)$ the set of all such metrics.

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We have that d_X and d_Y are *Euclidean*.

Idea: Let's force d to be Euclidean as well. We need to be precise about what we mean by a Euclidean metric.

Definition 1. *Let (Z, d) be a compact metric space. We say that the metric d is Euclidean if and only if there exist $d \in \mathbb{N}$ s.t. (Z, d) can be isometrically embedded into $\mathbb{R}^n$.*

For a finite metric space (Z, d), let $Z = \{z_1, \ldots, z_\ell\}$ and $D^{(2)}$ be the matrix with elements $d^2(z_i, z_j)$. Let $\mathbf{1}_\ell = (1, 1, \ldots, 1)^T \in \mathbb{R}^\ell$ and $\mathbf{I}_\ell$ be the $\ell \times \ell$ identity matrix. Let $Q_\ell = \mathbf{I}_\ell - \frac{1}{\ell}\mathbf{1}_\ell$. Consider the map $\tau_\ell : \mathbb{R}^{\ell\times\ell} \longrightarrow \mathbb{R}^{\ell\times\ell}$ given by $A \mapsto -\frac{1}{2} Q_\ell A Q_\ell$.

Proposition 1 (Blumenthal)**.** *A necessary and sufficient condition that a semi-metric space (Z, d), $\#Z = \ell$, be isometrically embeddable in some $\mathbb{R}^r$ $(r \in \mathbb{N})$ is that the matrix $\tau_\ell(D^{(2)})$ be positive semidefinite (PSD).*

In the case of a finite Euclidean metric space (Z, d), $Z = \{z_1, \ldots, z_\ell\}$, one says that the matrix $d(z_i, z_j)$ is a *Euclidean distance matrix* (**EDM**).

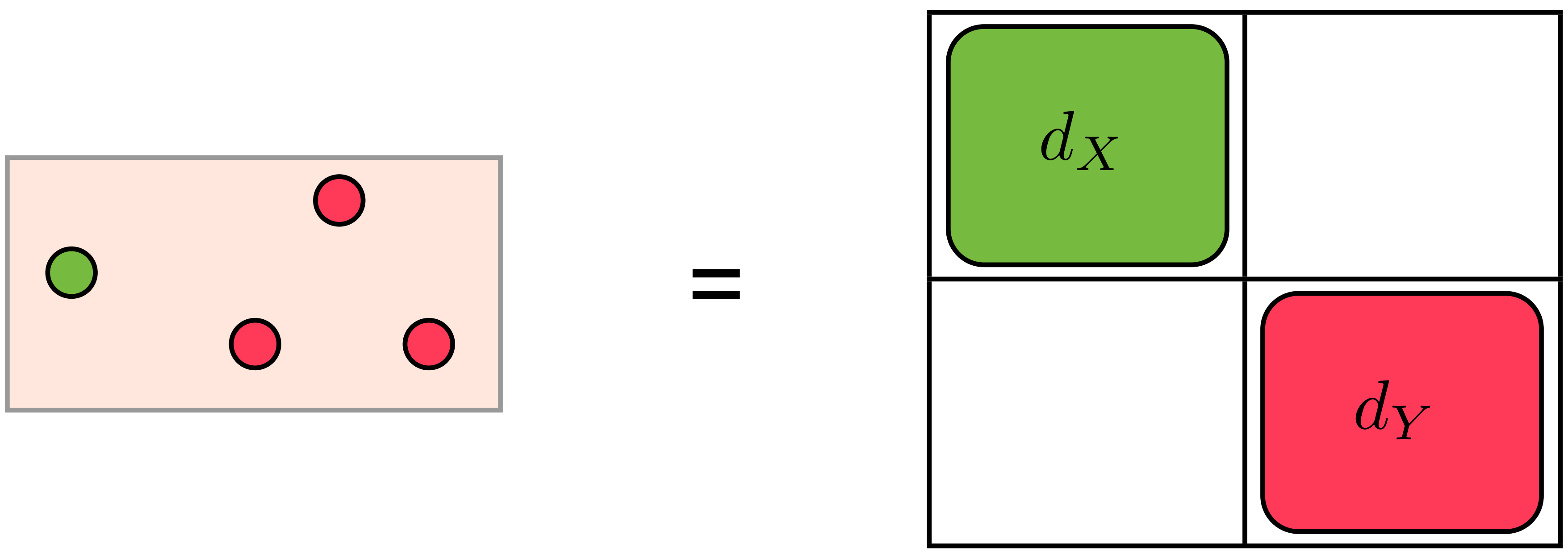
This gives us a direct *computational* way of checking whether a given distance matrix is an **EDM**.

For $X, Y \in \mathbb{R}^n$ let $\mathcal{D}_{\mathcal{E}}(X, Y)$ denote the set of metrics d on $X \sqcup Y$ such that $d(x, x') = \|x - x'\|$, $d(y, y') = \|y - y'\|$ for $x, x' \in X$ and $y, y' \in Y$, and d is *Euclidean.*

Let X and Y be *compact* subsets of $\mathbb{R}^n$ endowed with the Euclidean metric. Consider the following *tentative* distance

$$d_{\mathcal{GH}}^{E}(X, Y) := \inf_{d \in \mathcal{D}_{\mathcal{E}}(X,Y)} d_{\mathcal{H}}^{(X \sqcup Y, d)}(X, Y)$$

Theorem 1. *For* $X, Y \subset \mathbb{R}^n$ *compact,* $d_{\mathcal{H},iso}^{\mathbb{R}^n}(X, Y) = d_{\mathcal{GH}}^{E}(X, Y)$.



$\in D_E(d_X, d_Y)$

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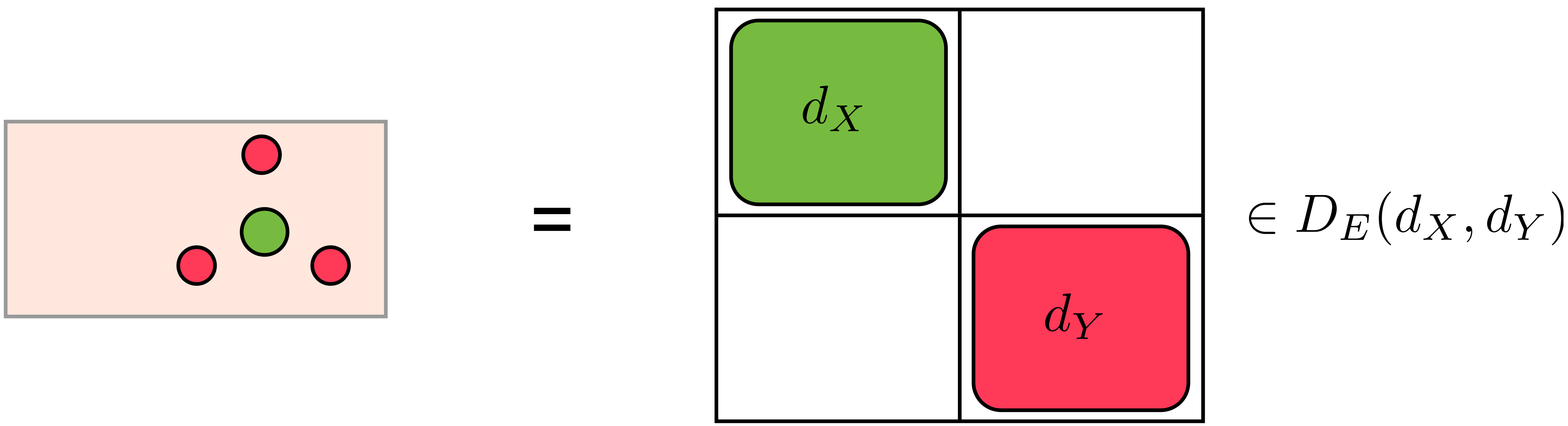

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$$d^E_{\mathcal{GH}}(X,Y) := \inf_{d \in \mathcal{D}_{\mathcal{E}}(X,Y)} d_{\mathcal{H}}^{(X \sqcup Y, d)}(X,Y)$$

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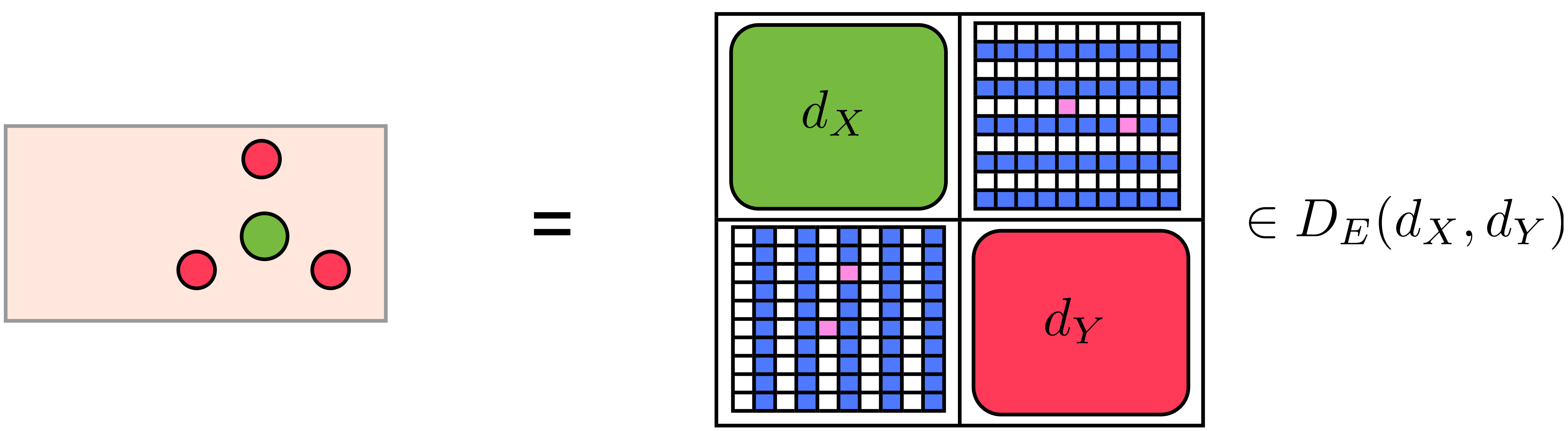
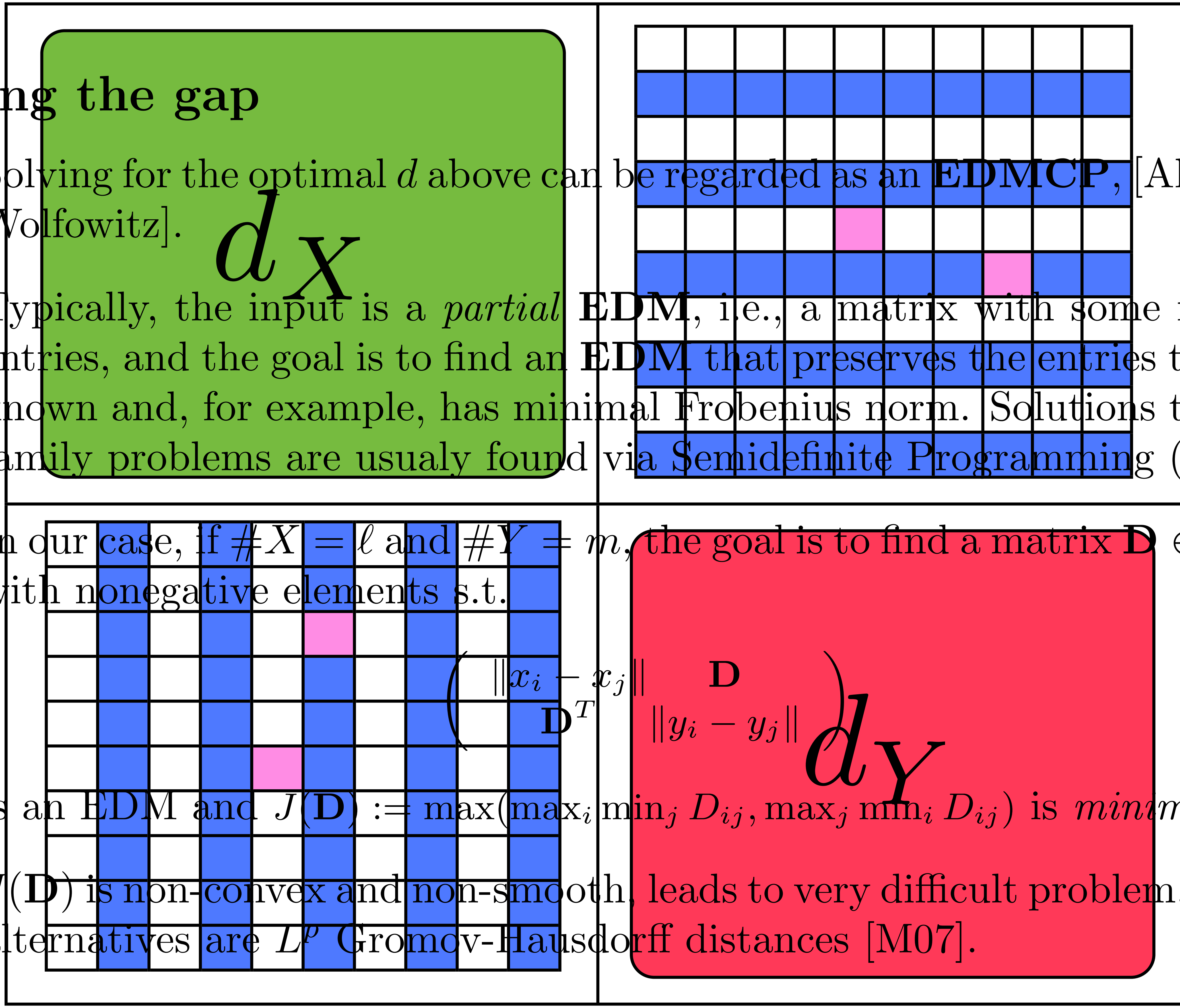


Closing the gap



- Solving for the optimal d above can be regarded as an **EDMCP**, [AlHomidan-Wolfowitz].

- Typically, the input is a *partial* **EDM**, i.e., a matrix with some missing entries, and the goal is to find an **EDM** that preserves the entries that are known and, for example, has minimal Frobenius norm. Solutions to these family problems are usualy found via Semidefinite Programming (SDP).

- In our case, if $\#X = \ell$ and $\#Y = m$, the goal is to find a matrix $\mathbf{D} \in \mathbb{R}^{\ell \times m}$ with nonegative elements s.t.

$$\begin{pmatrix} \|x_i - x_j\| & \mathbf{D} \\ \mathbf{D}^T & \|y_i - y_j\| \end{pmatrix}$$

  is an EDM and $J(\mathbf{D}) := \max(\max_i \min_j D_{ij}, \max_j \min_i D_{ij})$ is *minimized*.

- $J(\mathbf{D})$ is non-convex and non-smooth, leads to very difficult problem. Tractable alternatives are L^p Gromov-Hausdorff distances [M07].

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- Typically, the input is a *partial* **EDM**, i.e., a matrix with some missing entries, and the goal is to find an **EDM** that preserves the entries that are known and, for example, has minimal Frobenius norm. Solutions to these family problems are usualy found via Semidefinite Programming (SDP).

- In our case, if $\#X = \ell$ and $\#Y = m$, the goal is to find a matrix $\mathbf{D} \in \mathbb{R}^{\ell \times m}$ with nonegative elements s.t.

$$\begin{pmatrix} \|x_i - x_j\| & \mathbf{D} \\ \mathbf{D}^T & \|y_i - y_j\| \end{pmatrix}$$

  is an EDM and $J(\mathbf{D}) := \max(\max_i \min_j D_{ij}, \max_j \min_i D_{ij})$ is *minimized*.

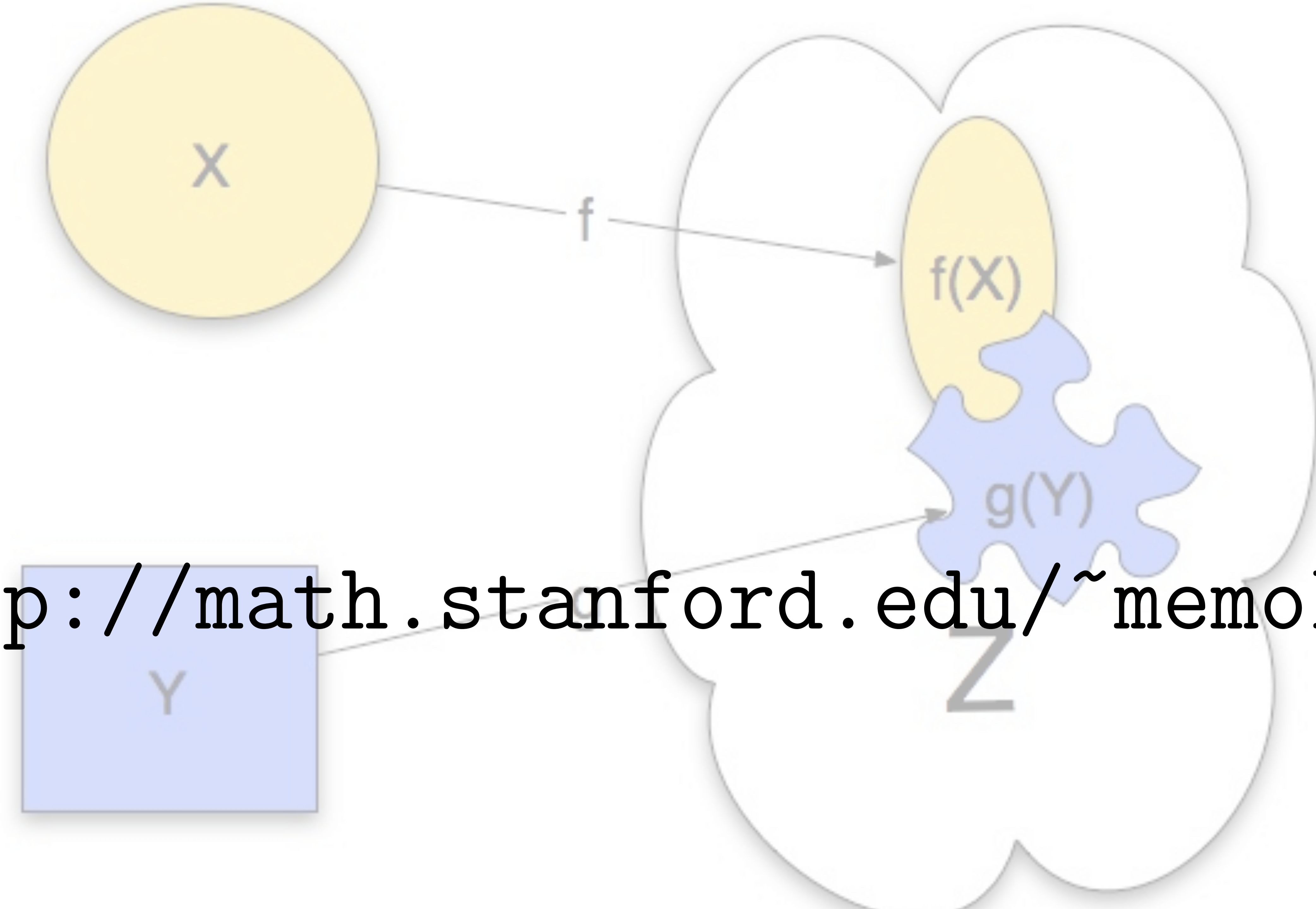
- $J(\mathbf{D})$ is non-convex and non-smooth, leads to very difficult problem. Tractable alternatives are L^p Gromov-Hausdorff distances [M07].

L^p- Gromov-Hausdorff distances in Euclidean spaces

- These distances [M07] provide a more general and more computationally tractable alternative than the standard **GH** distance.
- These distances are based on changing the Hausdorff part of the **GH** distance for the *Wasserstein* distance, a.k.a. Earth Mover's distance.
- In the context of Euclidean spaces, there are counterparts to **GH**, $d^E_{\mathcal{GH}}$ and **EH** distances.
- We've obtained a theoretical landscape parallel to that we've shown for **GH**.
- In the case of L^2 distances, the counterpart of $d^E_{\mathcal{GH}}$ we propose looking at is

$$\left(\min_{\mu,d} \sum_{x,y} d^2(x,y)\mu_{x,y} \right)^{1/2}$$

  where μ is a linearly constrained variable (a measure coupling) and $d \in \mathcal{D}_E(d_X, d_Y)$. Since d must be Euclidean, this can be expressed as a PSD condition on the matrix $\tau((d^2(x,y)))$ that can be dealt with easily.
- This optimization problem is substantially easier than it's **GH** counterpart.



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